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RADIATION RELAXATION TIMES AT HIGH TEMPERATURES

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RADIATION RELAXATION TIMES AT HIGH TEMPERATURES

by

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ABSTRACT

A time-dependent solution to the radiative transport equation is obtained which is valid for an optically thick medium. Its principal value is that it can be used to determine, for a given value of $\partial T/\partial t$ in the case of the energy density or $\partial\rho/\partial t$, $(\underset{\sim}{\nabla}\rho)/(\underset{\sim}{\nabla}T)$, and $\partial T/\partial t$ in the case of the flux, the approximate boundary of the region of matter temperature T and matter density ρ for which the radiation remains in local thermodynamic equilibrium with the matter. Numerical results for the radiation relaxation time, $c^{-1}\Lambda_t$, for hydrogen in the temperature range 1 kev $\leq$ kT $\leq$ 16 kev are obtained. These results can be transformed to apply to new temperatures or different elements. The upper range of validity of this method is thought to be $\rho \simeq 10^3$ g/cm^3 and kT $\simeq$ 64 kev. However, the transformation equations can be used to obtain a rough estimate of results for higher densities and temperatures. Although it might be expected that $\rho^2\Lambda_t$ would go as ρ in the low density limit where Compton scattering is expected to be dominant, it is found that for low densities $\rho^2\Lambda_t \rightarrow \text{const} \simeq \rho^2\Lambda_a/250$, where $\rho^2\Lambda_a$ is the high density value for $\rho^2\Lambda_t$ due to absorptive processes alone.

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CONTENTS

GLOSSARY OF NOTATION

$a = \dfrac{8\pi^5 k^4 T^4}{15 c^3 h^3}$	radiation constant, Eqs. (23) and (26)
a_n	expansion coefficient defined by Eq. (36)
A	constant defined by the first of Eqs. (49)
$b(kT)$	quantity defined by Eq. (59)
$B(\nu,T)$	Planck intensity, Eqs. (3)
$B(T) = \int_0^\infty B(\nu,T)d\nu = \dfrac{caT^4}{4\pi}$	Eqs. (22) and (23)
B	constant defined by the second of Eqs. (49)
c	velocity of light
E	electron kinetic energy in units of mc^2
E_m	matter energy density
$\mathcal{E}(\nu) = c^{-1}\int_\Omega I(\nu,\underset{\sim}{s})d\Omega$	energy density of radiation of frequency ν, Eqs. (2) and (21)
$\mathcal{E} = \int_0^\infty \mathcal{E}(\nu)d\nu$	total radiation energy density, Eq. (22)
$\mathcal{E}' = -4\pi c^{-2}\Lambda_t \partial B(T)/\partial t$	first order time-dependent correction to the radiation energy density
$\mathcal{E}(P,t_0)$	total radiation energy density at the specific point P and time t_0, Eqs. (61) and (63)
$\Delta\mathcal{E}(P,t_0)$	the part of $\mathcal{E}(P,t_0)$ coming from $x > x_0$ when no boundary exists at x_0, Eqs. (61) and (63)

$\Delta'\mathcal{E}(P,t_o)$	the part of $\mathcal{E}(P,t_o)$ coming from $x \geq x_o$ when account is taken of the boundary at x_o, Eq. (61)
$F_n(\nu) = \int_\Omega \underset{\sim}{n} \cdot \underset{\sim}{s} I(\nu,\underset{\sim}{s})d\Omega$	flux of radiation of frequency ν in direction $\underset{\sim}{n}$, Eqs. (1), (28), and (30)
$F_n = \int_0^\infty F_n(\nu)d\nu$	total radiation flux in direction $\underset{\sim}{n}$, Eqs. (29) and (35)
F'	first order time-dependent correction to the flux
$g_{ff}(Z_i^2/kT,u)$	free-free Gaunt factor
$g_{bfn}(Z_i^2/kT,u)$	bound-free Gaunt factor
h	Planck's constant
$I(\nu,\underset{\sim}{s})$	specific intensity of radiation of frequency ν traveling in direction $\underset{\sim}{s}$
k	Boltzmann's constant
K_c	to be consistent with the notation of other authors, we use K_c in this report to designate conductive opacity. Unfortunately, in S, we used K_c to designate opacity due to Compton scattering
$\ell(\nu)$	mean free path for photons of frequency ν
$\ell_r(\nu)$	coefficient in the solution for $I(\nu,\underset{\sim}{s})$ which is shown to be approximately equal to $\ell(\nu)$
$\ell_p(\nu)$	coefficient in the solution for $I(\nu,\underset{\sim}{s})$ which is shown to be approximately equal to $\ell(\nu)$
$\ell_q(\nu)$	coefficient in the solution for $I(\nu,\underset{\sim}{s})$ which is shown to be approximately equal to $\ell(\nu)$
$c^{-1}\ell_t(\nu)$	relaxation time for photons of frequency ν, also equal to the time equipartition of energy between photons of frequency ν and the matter

$\ell'(\nu) = \ell_t(\nu) + \ell(\nu)$	
$\ell^*(\nu)$	defined by Eq. (15)
ℓ	direction cosine, Eq. (S,18), Eqs. (37) to (42
m	mass of an electron
M_i	atomic weight of element i
$n(\nu,\underset{\sim}{s})$	occupation of the radiation oscillator corresponding to frequency ν and unit propagation vector $\underset{\sim}{s}$, Eq. (6)
$n(\nu,T)$	the occupation of the radiation oscillator corresponding to frequency ν when thermal equilibrium exists at temperature T, Eqs. (3)
n	expansion number, Eq. (36); also principal quantum number (57)
$\underset{\sim}{n}$	unit vector
N(P)	density of electrons with momentum $\underset{\sim}{P}$
$N = 2\pi\int_{-1}^{1}d\ell\int_{0}^{\infty}N(P)P^2dP$	density of electrons
P	magnitude of electron momentum; also a particular position
$\underset{\sim}{P}$	momentum of an electron
$q(E,\eta,T,u - u_2)$	degeneracy factor, Eq. (66)
$Q(\eta,T)$	corrective factor to account approximately for the effect of electron degeneracy on the Compton opacity, Eq. (67), Table 3
$\underset{\sim}{r}$	position relative to point P
$\underset{\sim}{s}$	unit vector in the direction of photon propagation

S	symbol denoting an earlier paper by the author (Sampson, 1959)
t	time
t_c	relaxation time for electrons, Eq. (5-26), page 78 of Spitzer (1956)
t_{eq}	time of equipartition of energy between electrons and ions, Eq. (5-31), page 80 of Spitzer (1956)
t_o	a specific value of time t
T	matter temperature
$T' = kT/mc^2$	
T_r	average radiation temperature, second of Eqs. (26)
$T_r(\nu)$	temperature of radiation of frequency ν, first of Eqs. (26)
$u = h\nu/kT$	reduced frequency
v	speed of an electron
$\underset{\sim}{V}$	local macroscopic velocity of the matter
$W(\underset{\sim}{r}, t_o + 3c^{-1}\Lambda_t)$	probability that a photon with random direction located at P at time t_o will be a distance $\underset{\sim}{r}$ from P at time $t_o + 3c^{-1}\Lambda_t$, Eq. (62)
x_o	distance along the x axis from P of the boundary of a region of appreciably different temperature, density, or composition
X_n	a quantity defined by Eq. (48)
$y = u\ell_t(\nu)$	Eq. (50)
Z_i	atomic number of element i
$\alpha = \eta + mc^2/kT$	1/kT times the relativistic chemical potential

α_i	mole fraction of element i
$\beta = v/c$	
$\gamma = h\nu/mc^2$	
$\delta(u,T)$	a second order term defined by Eq. (45)
∂	symbol indicating partial differentiation
η	1/kT times the nonrelativistic chemical potential for electrons
Θ	scattering angle, polar angle
$\lambda_o = \hbar/mc$	Compton wavelength
Λ	average photon mean free path given by the Rosseland mean of $\ell(\nu)$
Λ_a	Rosseland mean of $[\mu'_a(\nu)]^{-1}$
$\Lambda_c = 1/\rho K_c$	equivalent mean free path arising from electronic heat conduction
Λ_s	Rosseland mean of $[\mu_s(\nu)]^{-1}$
$c^{-1}\Lambda_t$	average photon relaxation time or relaxation time for the total radiation; also equal to the time of equipartition of energy between radiation and matter. It is given by the Rosseland mean of $c^{-1}\ell_t(\nu)$, Eq. (24)
$\mu'_a(\nu)$	absorptive contribution to $\ell_t(\nu)$ and $\ell(\nu)$. It is the usual absorption coefficient reduced by the factor [1 - exp(-u)] to take into account induced emission and is defined to exclude Compton scattering
$\mu'_{ff}(\nu)$	contribution to $\mu'_a(\nu)$ due to free-free transitions, Eq. (46)
$\mu'_{bf}(\nu)$	contribution to $\mu'_a(\nu)$ due to bound-free transitions, Eq. (57)
$\mu_r(\nu)$	quantity shown to be approximately equal to $\mu_s(\nu)$

$\mu_p(\nu)$	quantity shown to be approximately equal to $\mu_s(\nu)$
$\mu_q(\nu)$	quantity shown to be approximately equal to $\mu_s(\nu)$
$\mu_s(\nu)$	contribution to $\ell(\nu)^{-1}$ due to Compton scattering
$\mu_t(\nu)$	contribution to $\ell_t(\nu)$ due to Compton scattering
ν	frequency
ρ	matter density
σ	scattering cross section
σ_o	Thomson scattering cross section
Σ	summation sign
φ	azimuthal angle
Ω	solid angle

I. INTRODUCTION

We define local thermodynamic equilibrium to exist between matter and radiation of frequency ν at a point P whenever the vector flux and radiation energy density at P are given with sufficient accuracy by the equations

$$\underset{\sim}{F}(\nu) = -\frac{4\pi}{3}\,\ell(\nu)\underset{\sim}{\nabla}B(\nu,T) \tag{1}$$

and

$$\mathcal{E}(\nu) = \frac{4\pi}{c}\,B(\nu,T), \tag{2}$$

respectively, in which $\ell(\nu)$ is the photon mean free path, T is the temperature of the matter in the neighborhood of P, and $B(\nu,T)$ is the Planck intensity given by

$$\begin{aligned} B(\nu,T) &= \frac{2h\nu^3}{c^2}\,n(\nu,T); \\ n(\nu,T) &= \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}. \end{aligned} \tag{3}$$

Ordinarily, whether or not the radiation field is in local thermodynamic equilibrium with the matter is determined by whether or not the system is optically thick. In this report, we consider a different situation, one in which the system is optically thick, but the time rate of change of the matter temperature T is so great that the radiation field may not remain in thermal equilibrium with the matter. It is thought that the results obtained might be of particular interest in a detailed theoretical treatment of the most rapid stages of stellar evolution. For most material densities at very high temperatures, a large fraction of the energy density and energy flow is radiative. Thus any appreciable departure of the radiative flux or energy density from their equilibrium values will have a significant effect on the matter temperature and rate of energy production.

First we obtain an explicitly time-dependent solution to the equation of radiative transport. From this, we get first order time-dependent corrections to expressions (1) and (2) for the radiative flux and energy density. These results can be used in borderline cases to extend the conventional treatment of radiation. However, the range of conditions for which first order time-dependent corrections are both significantly large and valid is fairly narrow. Thus our principal purpose in obtaining these results is that they can be used for given values of the time and space derivatives of matter density ρ and matter temperature T to determine the approximate boundary of the regions of ρ and T for which the radiation stays in local thermodynamic equilibrium with the matter.

For the energy density of radiation of frequency ν, we find that this boundary is determined by a knowledge of $\ell_t(\nu)$ or for the total radiation by Λ_t. The quantity $c^{-1}\ell_t(\nu)$ and its Rosseland mean value $c^{-1}\Lambda_t$ can appropriately be called the relaxation times for photons of frequency ν and the total radiation, respectively. For the radiative flux, this boundary is a function of $\ell(\nu)$ as well as $\ell_t(\nu)$.

When scattering is not treated explicitly, it is a trivial matter to determine $\ell_t(\nu)$. It is simply equal to the photon mean free path $\ell(\nu)$. The motivation to carry out this work came when, by treating scattering explicitly, we obtained the integral equation [Eq. (9), section II] for $\mu_t(\nu)$, the scattering contribution to $\ell_t(\nu)^{-1}$. As a consequence of the omission of the $\cos\Theta$ factor which occurs in Eq. (10) for $\mu_s(\nu)$, the scattering contribution to $\ell(\nu)^{-1}$, it was immediately seen that $\mu_t(\nu) \ll \mu_s(\nu)$. Thus when $\mu_a'(\nu) \ll \mu_s(\nu)$, where $\mu_a'(\nu)$ is the absorptive contribution to $\ell_t^{-1}(\nu)$ and $\ell^{-1}(\nu)$, it is apparent from Eqs. (8) of section II that $\ell_t(\nu) \gg \ell(\nu)$. Since the condition $\mu_a'(\nu) \ll \mu_s(\nu)$ applies for most densities at high temperatures, it was thought that possibly at high temperatures the photon relaxation time $c^{-1}\ell_t(\nu)$ would be large enough that in some cases with large $\partial T/\partial t$ the radiation would not remain in local thermodynamic equilibrium with the matter.

Much of the theory and many of the equations we use are identical or similar to some of the theory and equations in an earlier paper (Sampson, 1959) in which the time-independent transport of radiation is considered and results for $\ell(\nu)$ are obtained when Compton scattering is dominant. We shall refer to this earlier paper as S, and equation n of S will be designated Eq. (S,n).

II. A TIME-DEPENDENT SOLUTION TO THE RADIATIVE TRANSPORT EQUATION

If one adds a term $c^{-1}\partial I(\nu,\underset{\sim}{s})/\partial t$ to the left-hand side of Eqs. (S,5) and (S,10), they are also valid for the time-dependent case. The latter equation can then be written

$$\frac{1}{c}\frac{\partial I(\nu,\underset{\sim}{s})}{\partial t} + \underset{\sim}{s}\cdot\underset{\sim}{\nabla} I(\nu,\underset{\sim}{s}) = -\mu_a'(\nu)\Big[I(\nu,\underset{\sim}{s}) - B(\nu,T)\Big]$$

$$- \int_{\underset{\sim}{P}}\int_{\Omega_2} N(P)\,d\underset{\sim}{P}\left[\frac{d\sigma(\nu,\underset{\sim}{s},\Theta,\underset{\sim}{P})}{d\Omega_2}\right] d\Omega_2\,\frac{2h\nu^3}{c^2}\left\{\begin{array}{l} n(\nu,\underset{\sim}{s})\Big[1 + n(\nu_2,\underset{\sim}{s}_2)\Big] \\ \\ -n(\nu_2,\underset{\sim}{s}_2)\Big[1 + n(\nu,\underset{\sim}{s})\Big]\exp(u_2 - u)\end{array}\right\} \tag{4}$$

where the reduced frequency u is defined by

$$u \equiv \frac{h\nu}{kT}, \tag{5}$$

and the occupation of the radiation oscillator corresponding to frequency ν and unit propagation vector $\underset{\sim}{s}$ is defined as

$$n(\nu,\underset{\sim}{s}) \equiv \frac{c^2}{2h\nu^3}\, I(\nu,\underset{\sim}{s}). \tag{6}$$

In Eq. (4), $\mu_a'(\nu)$ is the absorption coefficient multiplied by the factor $[1 - \exp(-u)]$ and defined to exclude Compton scattering, which is taken into account by the integral term; the subscript 2 refers to the final state of photons with initial intensity $I(\nu,\underset{\sim}{s})$ after being scattered through an angle θ by collisions with electrons of momentum $\underset{\sim}{P}$; $\left[d\sigma(\nu,\underset{\sim}{s},\theta,\underset{\sim}{P})/d\Omega_2\right]d\Omega_2$ is the differential scattering cross section; and $N(P)$ is the relativistic Maxwellian electron distribution function. The only assumptions made in writing the radiative transport equation in this form are that the matter has distribution functions characteristic of thermal equilibrium at temperature T, and electron degeneracy can be neglected.

Assuming the medium to be optically thick and the space and time derivatives of T and ρ to be sufficiently small, we can expand the equation for the intensity about the Planck distribution, Eqs. (3). In making the expansion, we choose terms of order $(n + 1)$, which involve derivatives of order $(n + 1)$, to be given by the nth order terms after they are acted upon by the operator $(c^{-1}\partial/\partial t + \underset{\sim}{s} \cdot \underset{\sim}{\nabla})$ and multiplied by unknown coefficients $\ell_i(\nu)$. Writing explicitly only those second order terms which give first order time-dependent corrections to the flux, we have

$$\begin{aligned} I(\nu,\underset{\sim}{s}) = {} & B(\nu,T) - \ell(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}B(\nu,T) - c^{-1}\ell_t(\nu)\,\frac{\partial B(\nu,T)}{\partial t} \\ & + c^{-1}\ell_r(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}_T\left\{\left[\ell_t(\nu) + \ell(\nu)\right]\frac{\partial B(\nu,T)}{\partial t}\right\} \\ & + c^{-1}\ell_p(\nu)\left[\frac{\partial \ell(\nu)}{\partial \rho}\right]\underset{\sim}{s} \cdot \underset{\sim}{\nabla}B(\nu,T)\,\frac{\partial \rho}{\partial t} \\ & + c^{-1}\ell_q(\nu)\left[\frac{\partial \ell_t(\nu)}{\partial \rho}\right]\left[\frac{\partial B(\nu,T)}{\partial t}\right]\underset{\sim}{s} \cdot \underset{\sim}{\nabla}\rho. \end{aligned} \tag{7}$$

By $\nabla_{\mathbf{T}}$ acting on a quantity, we mean that we determine the space gradient of the quantity only through its dependence on T or the time derivative of T, i.e., we hold ρ constant. The last three terms in Eq. (7) arise from the quantities $c^{-1}\partial/\partial t\left[\ell(\nu)\mathbf{s}\cdot\nabla B(\nu,T)\right]$ and $c^{-1}\mathbf{s}\cdot\nabla\left[\ell_t(\nu)\partial B(\nu,T)/\partial t\right]$ after they have been rearranged as three separate terms proportional to $(\mathbf{s}\cdot\nabla T)(\partial T/\partial t)$, $(\mathbf{s}\cdot\nabla T)(\partial\rho/\partial t)$, and $(\mathbf{s}\cdot\nabla\rho)(\partial T/\partial t)$, respectively, each one multiplied by a different coefficient.

When we substitute Eq. (7) into Eq. (4), the zero order terms cancel on the right-hand side, and we find by similar arithmetic to that used in S

$$
\begin{aligned}
\left[\ell(\nu)\right]^{-1} &= \mu_a'(\nu) + \mu_s(\nu);\\
\left[\ell_t(\nu)\right]^{-1} &= \mu_a'(\nu) + \mu_t(\nu);\\
\left[\ell_r(\nu)\right]^{-1} &= \mu_a'(\nu) + \mu_r(\nu);
\end{aligned}
\tag{8}
$$

etc. In these equations,

$$\mu_t(\nu) = \int_{\mathbf{P}}\int_{\Omega_2} N(P)d\mathbf{P}\,\frac{d\sigma}{d\Omega_2}\,d\Omega_2\left[\frac{1-\exp(-u)}{1-\exp(-u_2)}\right]\left[1-\frac{\nu_2\ell_t(\nu_2)}{\nu\ell_t(\nu)}\right], \tag{9}$$

and $\mu_s(\nu)$ is given by the same expression if the quantities in the brackets are replaced with[1]

---

[1] This is the same result for $\mu_s(\nu)$ as was obtained in S; however, Eq. (S,16) is not quite logically correct, because one should make approximations (S,A) and (S,B) before replacing $(\nu_2\mathbf{s}_2\cdot\nabla T)(\nu\mathbf{s}\cdot\nabla T)^{-1}$ with $\nu_2(\nu)^{-1}\cos\theta$.

$$\left[\frac{1 - \exp(-u)}{1 - \exp(-u_2)}\right]\left[1 - \frac{\nu_2 \ell(\nu_2)\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}T}{\nu \ell(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}T}\right] \simeq 1 - \frac{\nu_2}{\nu}\cos\theta. \qquad (10)$$

In writing the expression on the right-hand side, we have made approximations (S,A) and (S,B) and have used the result obtained in the appendix of S that $(\nu_2\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}T)(\nu\underset{\sim}{s} \cdot \underset{\sim}{\nabla}T)^{-1}$ has an effective value $\nu_2(\nu)^{-1}\cos\theta$.

The expressions for $\mu_r(\nu)$, $\mu_p(\nu)$, and $\mu_q(\nu)$ are only slightly more difficult to obtain. Neglecting for the moment second order terms involving products of first order derivatives of $B(\nu,T)$ or $n(\nu,T)$, we obtain the following equation in $\ell_r(\nu)$ analogous to Eq. (S,14) in $\ell(\nu)$:

$$\begin{aligned}
\underset{\sim}{s} \cdot \underset{\sim}{\nabla}_T\left[\ell'(\nu)\,\frac{\partial B(\nu,T)}{\partial t}\right] &= \mu_a'(\nu)\ell_r(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}_T\left[\ell'(\nu)\,\frac{\partial B(\nu,T)}{\partial t}\right] \\
&+ \ell_r(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}_T\left[\ell'(\nu)\,\frac{\partial B(\nu,T)}{\partial t}\right]\int_{\underset{\sim}{P}}\int_{\Omega_2} N(P)d\underset{\sim}{P}\,\frac{d\sigma}{d\Omega_2}\,d\Omega_2 \\
&\cdot\left\{\begin{array}{l} 1 + n(\nu_2,T)\left[1 - \exp(u_2 - u)\right] \\ - \dfrac{\ell_r(\nu_2)\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}_T\left[\ell'(\nu_2)\partial n(\nu_2,T)/\partial t\right]}{\ell_r(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}_T\left[\ell'(\nu)\partial n(\nu,T)/\partial t\right]}\left[-n(\nu,T) + \left\{1 + n(\nu,T)\right\}\exp(u_2 - u)\right]\end{array}\right\}, \qquad (11)
\end{aligned}$$

where

$$\ell'(\nu) = \ell_t(\nu) + \ell(\nu). \qquad (12)$$

Assuming we can write $T = g(x,y,z)f(t)$, it is apparent that

$$\underset{\sim}{\nabla}\left(\frac{\partial T}{\partial t}\right) = T^{-1}(\underset{\sim}{\nabla}T)\,\frac{\partial T}{\partial t}\,. \qquad (13)$$

Then using Eqs. (3) and (5), we obtain

$$\underset{\sim}{s} \cdot \underset{\sim}{\nabla}_T\left[\ell'(\nu)\,\frac{\partial n(\nu,T)}{\partial t}\right] = \frac{u}{T^2}\,\frac{\exp(u)}{[\exp(u) - 1]^2}\,\ell^*(\nu)(\underset{\sim}{s} \cdot \underset{\sim}{\nabla}T)\frac{\partial T}{\partial t}\,, \qquad (14)$$

in which

$$\ell^*(\nu) = T\,\frac{\partial \ell'(\nu)}{\partial T} + \ell'(\nu)\left\{u\left[\frac{\exp(u) + 1}{\exp(u) - 1}\right] - 1\right\}. \qquad (15)$$

Substituting this result into Eq. (11) and again using the second of Eqs. (3), we can reduce the part in the large braces to

$$\frac{1 - \exp(-u)}{1 - \exp(-u_2)}\left\{1 - \left[\frac{\nu_2\ell_r(\nu_2)\ell^*(\nu_2)}{\nu\ell_r(\nu)\ell^*(\nu)}\right]\left[\frac{\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}T}{\underset{\sim}{s} \cdot \underset{\sim}{\nabla}T}\right]\right\}. \qquad (16)$$

Then solving Eq. (11) for $\ell_r(\nu)$ and using Eqs. (8) and (16), we get the same result for $\mu_r(\nu)$ as was obtained for $\mu_s(\nu)$ if $\ell^*(\nu_2)/\ell^*(\nu)$ is replaced with unity. Similarly for $\mu_p(\nu)$ and $\mu_q(\nu)$, we get the same result as we got for $\mu_s(\nu)$ except that $\left[\ell(\nu_2)\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}T\right]/\left[\ell(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}T\right]$ in Eq. (10) is replaced with

$$\left[\frac{\ell_p(\nu_2)\partial\ell(\nu_2)/\partial\rho}{\ell_p(\nu)\partial\ell(\nu)/\partial\rho}\right]\frac{\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}T}{\underset{\sim}{s} \cdot \underset{\sim}{\nabla}T}$$

and

$$\left[\frac{\ell_q(\nu_2)\partial\ell_t(\nu_2)/\partial\rho}{\ell_q(\nu)\partial\ell_t(\nu)/\partial\rho}\right]\frac{\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla}\rho}{\underset{\sim}{s} \cdot \underset{\sim}{\nabla}\rho}\,,$$

respectively. Since the theory given in the appendix of S is obviously equally applicable if $\underset{\sim}{\nabla}T$ is replaced with $\underset{\sim}{\nabla}\rho$, the approximations

$$\frac{\ell^*(\nu_2)}{\ell^*(\nu)} \simeq \frac{\partial\ell(\nu_2)/\partial\rho}{\partial\ell(\nu)/\partial\rho} \simeq \frac{\partial\ell_t(\nu_2)/\partial\rho}{\partial\ell_t(\nu)/\partial\rho} \simeq 1 \tag{17}$$

lead to

$$\ell_r(\nu) \simeq \ell_p(\nu) \simeq \ell_q(\nu) \simeq \ell(\nu). \tag{18}$$

When these approximations are made, Eq. (7) for the intensity can be rewritten as

$$I(\nu,\underset{\sim}{s}) = B(\nu,T) - c^{-1}\ell_t(\nu)\frac{\partial B(\nu,T)}{\partial t} - \ell(\nu)\underset{\sim}{s}\cdot\underset{\sim}{\nabla}\left[B(\nu,T) - c^{-1}\ell_t(\nu)\frac{\partial B(\nu,T)}{\partial t}\right]$$
$$+ c^{-1}\ell(\nu)\frac{\partial}{\partial t}\left[\ell(\nu)\underset{\sim}{s}\cdot\underset{\sim}{\nabla}B(\nu,T)\right] + \dots \,. \tag{19}$$

For temperatures below the kilovolt region, approximations (17) are very good because $\nu \simeq \nu_2$. For higher temperatures, they become increasingly worse as temperature is increased. Even at very high temperatures, they are still good if $\ell^*(\nu)$, $\partial\ell(\nu)/\partial\rho$, and $\partial\ell_t(\nu)/\partial\rho$ vary slowly with frequency. It turns out that this is the case (with the exception of a few low frequencies unimportant in determining frequency averages) for the densities at which we are able to obtain results for $\ell_t(\nu)$.

Let us now consider the effect of neglecting second order terms of the type $\ell(\nu)\ell_t(\nu_2)\left[\underset{\sim}{s}\cdot\underset{\sim}{\nabla}B(\nu,T)\right]\left[\partial B(\nu_2,T)/\partial t\right]$. No terms of exactly this form appear on the left-hand side of Eq. (4) when Eq. (7) is substituted for $I(\nu,\underset{\sim}{s})$; however, these terms are proportional to $(\underset{\sim}{s}\cdot\underset{\sim}{\nabla}T)(\partial T/\partial t)$, so they should obviously be included in the equations involving $\ell_r(\nu)$. This adds to the part in the large braces in Eq. (11) a term

$$\frac{\left\{\ell_t(\nu_2)\ell(\nu)\left[\underset{\sim}{s} \cdot \underset{\sim}{\nabla} n(\nu,T)\right]\partial n(\nu_2,T)/\partial t + \ell_t(\nu)\ell(\nu_2)\left[\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla} n(\nu,T)\right]\partial n(\nu,T)/\partial t\right\}}{\ell_r(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}\left[\ell'(\nu)\partial n(\nu,T)/\partial t\right]}$$

$$\cdot \left[1 - \exp(u_2 - u)\right],$$

which we must show is small compared with unity, the approximate value of the part which has already been included. We rewrite this with the use of Eqs. (3), (5), and (14) as

$$\frac{\ell_t(\nu_2)\ell(\nu)}{\ell^*(\nu)\ell_r(\nu)}\left[1 + \frac{\ell_t(\nu)\ell(\nu_2)\underset{\sim}{s}_2 \cdot \underset{\sim}{\nabla} T}{\ell_t(\nu_2)\ell(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla} T}\right]\frac{u_2 \exp(u_2)}{\left[\exp(u_2) - 1\right]^2}\left[1 - \exp(u_2 - u)\right]$$

$$\simeq \frac{\ell_t(\nu_2)}{\ell^*(\nu)}\frac{u_2 \exp(u_2)}{\left[\exp(u_2) - 1\right]^2}\left\{1 - \exp(u_2 - u)\right\}, \qquad (20)$$

where in writing the right-hand side, we used approximations (18) and have neglected the second term in the large brackets on the left-hand side, because it goes approximately as $\cos\theta$ and thus has an effective value near zero. Noting that $\partial\ell'(\nu)/\partial T$ is always positive and that the quantity in the braces of Eq. (15) is equal to or greater than unity, we see from Eqs. (12) and (15) that $\ell_t(\nu)$ is always smaller (usually much smaller) than $\ell^*(\nu)$. For large u, the remaining part on the right-hand side of Eq. 20 has the small value $u_2\left[\exp(-u_2) - \exp(-u)\right]$, whereas for small u, it has the value $\left[u/u_2 - 1\right]$, which, according to the analysis of section IV, is about kT/mc^2 when the integrations over angles and

and electron energies are performed. The part in the braces of Eq. (20) is generally small enough that the whole quantity is also small for intermediate u, i.e., $u \sim 1$. Thus we conclude that neglecting these terms is valid for all frequencies until very high temperatures are reached.

III. TIME-DEPENDENT EXPRESSIONS FOR RADIATIVE ENERGY DENSITY AND FLUX

Multiplying Eq. (19) by c^{-1} and integrating over solid angle and frequency, we get

$$\mathcal{E}(\nu) = c^{-1} \int_{\Omega} I(\nu,\underset{\sim}{s})d\Omega = \frac{4\pi}{c}\left[B(\nu,T) - c^{-1}\ell_t(\nu)\frac{\partial B(\nu,T)}{\partial t}\right] \tag{21}$$

and

$$\mathcal{E} = \int_0^{\infty} \mathcal{E}(\nu)d\nu = \frac{4\pi}{c}\left[B(T) - c^{-1}\Lambda_t \frac{\partial B(T)}{\partial t}\right] \tag{22}$$

for the radiative energy density of frequency ν and the total radiative energy density, respectively. Here

$$B(T) = \int_0^{\infty} B(\nu,T)d\nu = \frac{caT^4}{4\pi}, \tag{23}$$

and Λ_t is the Rosseland mean value of $\ell_t(\nu)$:

$$\Lambda_t = \frac{15}{4\pi^4}\int_0^{\infty} \ell_t(\nu)u^4 \exp(u)\left[\exp(u) - 1\right]^{-2}du. \tag{24}$$

Using the approximations $\partial\mathcal{E}(\nu)/\partial t \simeq 4\pi c^{-1}\partial B(\nu,T)/\partial t$ and $\partial\mathcal{E}/\partial t \simeq 4\pi c^{-1}\partial B(T)/\partial t$, which are valid if the form given by Eqs. (7) and (19) for the intensity is valid, we see from Eqs. (21) and (22) that $c^{-1}\ell_t(\nu)$ and $c^{-1}\Lambda_t$ can appropriately be called the relaxation times for photons of frequency ν and the total radiation, respectively. With these approximations, we can rewrite Eqs. (21) and (22) as

$$\begin{aligned}\frac{\partial\mathcal{E}(\nu)}{\partial t} &= \frac{4\pi c^{-1}B(\nu,T) - \mathcal{E}(\nu)}{c^{-1}\ell_t(\nu)}\,,\\ \frac{\partial\mathcal{E}}{\partial t} &= \frac{4\pi c^{-1}B(T) - \mathcal{E}}{c^{-1}\Lambda_t}\,.\end{aligned}\tag{25}$$

Defining the radiation temperatures $T_r(\nu)$ and T_r by the equations

$$\begin{aligned}\mathcal{E}(\nu) &= \frac{8\pi h\nu^3}{c^3}\left\{-1 + \exp\left[\frac{h\nu}{kT_r(\nu)}\right]\right\}^{-1},\\ \mathcal{E} &= aT_r^4\,,\end{aligned}\tag{26}$$

we get for small $T - T_r(\nu)$ and $T - T_r$

$$\begin{aligned}\frac{\partial T_r(\nu)}{\partial t} &= \frac{\left[T - T_r(\nu)\right]}{c^{-1}\ell_t(\nu)}\,,\\ \frac{\partial T_r}{\partial t} &= \frac{(T - T_r)}{c^{-1}\Lambda_t}\,.\end{aligned}\tag{27}$$

From Eqs. (27), we see that $c^{-1}\ell_t(\nu)$ and $c^{-1}\Lambda_t$ are the times of equipartition of energy between matter and radiation of frequency ν and total radiation, respectively, analogous to the t_{eq} of Spitzer [1956, page 80, Eqs. (5-30) and (5-31)], in which t_{eq} is the time of equipartition of energy between electrons and ions. The fact that energy equipartition times and relaxation times for the radiation are equal arises from the fact that the photons interact only with the matter, not with each other.[2]

Again using Eq. (19), we get for the flux of frequency ν across a surface with unit normal $\underset{\sim}{n}$

$$F_n(\nu) = \int_\Omega \underset{\sim}{n} \cdot \underset{\sim}{s} I(\nu, \underset{\sim}{s}) d\Omega$$

$$= -\frac{4\pi}{3} \ell(\nu) \left\{ \underset{\sim}{n} \cdot \underset{\sim}{\nabla} \left[B(\nu,T) - c^{-1}\ell_t(\nu) \frac{\partial B(\nu,T)}{\partial t} \right] - \frac{\partial}{\partial t} \left[c^{-1}\ell(\nu) \underset{\sim}{n} \cdot \underset{\sim}{\nabla} B(\nu,T) \right] \right\}. \qquad (28)$$

When $\mu_s(\nu) \ll \mu_a'(\nu)$, we see from Eqs. (8) that $\ell(\nu) \simeq \mu_s(\nu)^{-1}$. Then, as seen from the results in S, $\partial\ell(\nu)/\partial\rho \simeq -\rho^{-1}\ell(\nu)$ and $\partial\ell(\nu)/\partial T \simeq 0$. Furthermore, $\ell(\nu)$ is a very slowly varying function of ν. Hence it is a good approximation in integrating Eq. (28) over frequency to remove $\ell(\nu)$ from the integration of the second and third terms and replace it with

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[2] In this paper, "relaxation time" refers only to the time it takes quantities of a given kind to reach their characteristic thermal distribution. For ions or electrons, this time is the t_c of Spitzer (1956, p. 78).

its Rosseland mean, Λ. We obtain

$$
\begin{aligned}
F_n &= \int_0^\infty F_n(\nu)d\nu \\
&\simeq -\frac{4\pi}{3}\left\{\Lambda\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\left[B(T) - c^{-1}\Lambda_t\frac{\partial B(T)}{\partial t}\right]\right. \\
&\qquad \left. - c^{-1}\Lambda^2\left[4 - \frac{\rho^{-1}\partial\rho/\partial t}{T^{-1}\partial T/\partial t}\right]T^{-1}\frac{\partial T}{\partial t}\underset{\sim}{n}\cdot\underset{\sim}{\nabla}B(T)\right\} ; \\
&\frac{\Lambda_a}{\Lambda_s} \gg 1,
\end{aligned}
\tag{29}
$$

where Λ_s and Λ_a are the Rosseland mean values of $\left[\mu_s(\nu)\right]^{-1}$ and $\left[\mu_a'(\nu)\right]^{-1}$, respectively. In obtaining the term proportional to Λ^2, we used Eqs. (13) and (23). For densities a little lower than the highest for which Eq. (29) applies, one can replace Λ everywhere with Λ_s. Also unless $\left|(\rho^{-1}\partial\rho/\partial t)/(T^{-1}\partial T/\partial t)\right| \gg 1$, we can neglect the term proportional to Λ^2 because, as noted in the latter part of the Introduction, $\ell_t(\nu) \gg \ell(\nu)$ when $\mu_s(\nu) \gg \mu_a'(\nu)$. Under these conditions, one can also neglect the last term in Eqs. (19) and (28) for $I(\nu,\underset{\sim}{s})$ and $F_n(\nu)$. Then we have the same results for radiative intensity, energy density, and flux as in the time-independent case, with the exception that $B(\nu,T)$ and $B(T)$ have everywhere been replaced with $\left[B(\nu,T) - c^{-1}\ell_t\partial B(\nu,T)/\partial t\right]$ and $\left[B(T) - c^{-1}\Lambda_t\partial B(T)/\partial t\right]$, respectively.

For higher densities and lower temperatures, both $\mu_s(\nu)$ and $\mu_a'(\nu)$ are significant in determining $\ell(\nu)$; however, $\mu_t(\nu)$ can usually be neglected in determining $\ell_t(\nu)$, i.e., $\ell_t(\nu) \simeq \left[\mu_a'(\nu)\right]^{-1}$. Before proceeding with the derivation of the approximate expression for the total flux which is valid in this case, it is convenient to rewrite Eq. (28) in the following way:

$$F_n(\nu) = -\frac{4\pi}{3}\,\ell(\nu)\underset{\sim}{n}\cdot\underset{\sim}{\nabla}B(\nu,T)$$
$$\cdot\left\{1-\left(\begin{array}{l}\left[\ell_t(\nu)+\ell(\nu)\right]\left\{u\left[\dfrac{\exp(u)+1}{\exp(u)-1}\right]-1\right\}+T\dfrac{\partial}{\partial T}\left[\ell_t(\nu)+\ell(\nu)\right]\\[2ex] +\left[\rho\,\dfrac{\partial\ell_t(\nu)}{\partial\rho}\,\dfrac{T\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\rho}{\rho\underset{\sim}{n}\cdot\underset{\sim}{\nabla}T}+\rho\,\dfrac{\partial\ell(\nu)}{\partial\rho}\,\dfrac{T\partial\rho/\partial t}{\rho\partial T/\partial t}\right]\end{array}\right)\frac{T^{-1}}{c}\,\frac{\partial T}{\partial t}\right\}, \tag{30}$$

where we have again used Eq. (13). We also used the relation

$$\frac{\partial^2 B(\nu,T)}{\partial T^2} = T^{-1}\left[u\left(\frac{\exp(u)+1}{\exp(u)-1}\right)\frac{\partial B(\nu,T)}{\partial T}\right]\ , \tag{31}$$

which is easily derived from Eqs. (3) and (5). We assume that $\mu_a'(\nu)$ is the free-free absorption coefficient given by Eq. (46). (See section IV.) Then we have

$$\rho\,\frac{\partial\ell_t(\nu)}{\partial\rho} \simeq -2\ell_a(\nu); \qquad \frac{\mu_t(\nu)}{\mu_a'(\nu)} \ll 1 \tag{32}$$

and

$$T\frac{\partial}{\partial T}\left[\ell_t(\nu)+\ell(\nu)\right] \simeq T\frac{\partial}{\partial T}\left[\ell_a(\nu)+\ell(\nu)\right] \simeq 0; \qquad \frac{\mu_t(\nu)}{\mu_a'(\nu)} \ll 1. \tag{33}$$

Also

$$\rho \frac{\partial \ell(\nu)}{\partial \rho} = - \frac{1}{\left[\mu_s(\nu) + \mu_a'(\nu)\right]^2} \left[\mu_s(\nu) + 2\mu_a(\nu)\right] = -\ell(\nu) - \frac{\ell^2(\nu)}{\ell_a(\nu)} \, , \qquad (34)$$

where $\ell_a \equiv \left[\mu_a'(\nu)\right]^{-1}$. Approximation (33) may at first seem invalid if one is used to seeing $\ell_a(\nu)$ written as a function of u rather than ν as in Eq. (46); however, in the case of hydrogen, approximate values for $\ell_a(\nu)^{-1} T \partial \ell_a(\nu)/\partial T$ at $u = 8$ are 0.28 and 0.23 for temperatures of 0.25 and 1 kev, respectively. This gives a contribution only about 2 or 2.5% of the contribution of the first term in the large parentheses of Eq. (35). Substituting these results into Eq. (30), we have an expression which is a function of $\ell(\nu)$ and $\ell_a(\nu)$. These quantities are generally known quite well from opacity work. Integrating over frequency, we obtain

$$F_n = - \frac{4\pi}{3} \Lambda \underset{\sim}{n} \cdot \underset{\sim}{\nabla} B(T)$$

$$\cdot \left\{ 1 - \Lambda^{-1} \left(\overline{\ell(\nu)\left[\ell_a(\nu) + \ell(\nu)\right]\left[u \frac{\exp(u) + 1}{\exp(u) - 1} - 1\right]} - 2\overline{\ell(\nu)\ell_a}(\nu) \frac{T\underset{\sim}{n} \cdot \underset{\sim}{\nabla}\rho}{\rho\underset{\sim}{n} \cdot \underset{\sim}{\nabla}T} \right. \right.$$

$$\left. \left. - \overline{\left[\ell^2(\nu) + \frac{\ell^3(\nu)}{\ell_a(\nu)}\right]} \frac{T\partial\rho/\partial t}{\rho \partial T/\partial t} \right) \frac{T^{-1}}{c} \frac{\partial T}{\partial t} \right\} ; \qquad \frac{\Lambda_t}{\Lambda_a} \simeq 1, \qquad (35)$$

in which a bar over a quantity means that the Rosseland mean of that quantity has been taken. At very high densities, $\mu_s(\nu)$ is negligible, and $\ell(\nu)$ can be replaced with $\ell_a(\nu)$ everywhere in Eq. (35). If the terms in the large parentheses in Eq. (35) nearly cancel, approximation (33) should not be made.

IV. METHODS OF CALCULATING THE RELAXATION TIMES $c^{-1}\ell_t(\nu)$ AND $c^{-1}\Lambda_t$

Solutions to Eqs. (8) and (9) for $\ell_t(\nu)$ are obviously very difficult to obtain in general. Here we obtain an approximate solution by fitting $\ell_t(\nu)$ to a polynomial and expanding in powers of $\gamma = h\nu/mc^2$ and $\beta = v/c$, where v is the electron speed. We assume

$$u\ell_t(\nu) = \sum_n a_n u^n \tag{36}$$

and substitute this expression for $u\ell_t(\nu)$ into Eq. (9). As in S, we find it convenient to perform the integration over solid angle $d\Omega_2$ in the primed system in which the electron is at rest, because in that system the differential scattering cross section is axially symmetric. The relationship between cross sections in primed and unprimed systems is given by (S,19). Using that equation and choosing $d\Omega_2'$ and $d\underset{\sim}{P}$ as in Eqs. (S,22) and (S,24), we get

$$\mu_t(\nu) = \frac{4\pi}{u\ell_t(\nu)} \sum a_n u^n \int_0^\infty N(P)P^2 dP \int_{-1}^{1} \frac{d\ell}{2} \int_0^{2\pi} d\varphi' \int_{-1}^{1} (1 - \beta\ell) d(\cos\theta')$$
$$\cdot \left[1 - \left(\frac{u_2}{u}\right)^n\right] \frac{d\sigma'}{d\Omega_2'} , \tag{37}$$

where approximation (S,A), $\left[1 - \exp(-u)\right]\left[1 - \exp(-u_2)\right]^{-1} \simeq 1$, has again been made. Making use of the well-known relation between initial and scattered frequencies in the primed system,

$$\frac{\nu_2'}{\nu'} = \frac{1}{1 + \gamma'(1 - \cos\theta')} , \qquad (38)$$

together with Eqs. (S,18) relating frequencies and direction cosines in primed and unprimed systems, one obtains

$$\left(\frac{u_2}{u}\right)^n = \frac{(1 + \beta\ell_2')^n(1 - \beta\ell)^n}{(1 - \beta^2)^n\left[1 + \gamma'(1 - \cos\theta')\right]^n} , \qquad (39)$$

in which

$$\gamma' = \frac{h\nu'}{mc^2} = (1 - \beta\ell)(1 - \beta^2)^{-1/2}\gamma. \qquad (40)$$

Combining the last of Eqs. (S,18) with the first of Eqs. (S,21), we have

$$\ell_2' = \frac{\ell - \beta}{1 - \beta\ell}\cos\theta' + \sin\theta'\left[1 - \left(\frac{\ell - \beta}{1 - \beta\ell}\right)^2\right]^{1/2}\cos(\varphi_x' - \varphi'). \qquad (41)$$

We substitute the Klein-Nishina formula, Eq. (S,17), for $d\sigma'/d\Omega_2'$ into Eq. (37); use Eqs. (39), (40), and (41) to express all angular functions in terms of ℓ, $\cos\theta'$, and φ'; and make an expansion in powers of β and γ. After integrating over φ', $\cos\theta'$, and ℓ and performing some tedious algebra, we get the following result to order β^4, $\beta^2\gamma$, and γ^2:

$$\int_{-1}^{1}\frac{d\ell}{2}\int_{0}^{2\pi}d\varphi'\int_{-1}^{1}(1-\beta\ell)d(\cos\theta')\frac{d\sigma'}{d\Omega_2'}\left[1-\left(\frac{u_2}{u}\right)^n\right]$$

$$= n\sigma_o\left\{\left[\gamma-(3+n)\frac{\beta^2}{3}\right]-\frac{1}{10}\left[(7n+35)\gamma^2-(14n^2+84m+137)\frac{\beta^2}{3}\gamma\right.\right.$$

$$\left.\left.+(7n^3+42n^2+85n+66)\frac{\beta^4}{15}\right]\right\}, \qquad (42)$$

in which σ_o is the Thomson cross section $(8\pi/3)(e^2/mc^2)^2$. The only tricky part in obtaining Eq. (42) is in the evaluation of the contribution of various powers of $\beta\ell_2'$ arising from the expansion of Eq. (39). Hence we will give the results for those quantities. Using Eq. (41), we find after integration over φ' that $\beta\ell_2'$ and $(\beta\,\ell_2')^2$ have the values

$$\beta\ell_2' = \frac{\beta(\ell-\beta)}{1-\beta\ell}\cos\theta';$$

$$(\beta\ell_2')^2 = \frac{\beta^2(\ell-\beta)^2}{(1-\beta\ell)^2}\left(\frac{3\cos\theta'-1}{2}\right)+\frac{(1-\cos^2\theta)}{2}\beta^2,$$

whereas $(\beta\ell_2')^3$ gives no contribution to order β^4 because it is odd in $\cos\theta'$. After performing quite a bit of arithmetic and integrating over $\cos\theta'$ and ℓ, as well as φ', one finds that the effective value of $(\beta\ell_2')^4$ is simply $\beta^4\ell^4$ to order β^4.

Using a relativistic Maxwellian distribution function for N(P), we find

$$\frac{\overline{\beta^2}}{3} = \frac{4\pi}{N} \int_0^\infty \frac{\beta^2}{3} N(P)P^2 dP = \frac{kT}{mc^2} - \frac{5}{2}\left(\frac{kT}{mc^2}\right)^2 + \dots;$$

$$\frac{\overline{\beta^4}}{15} = \left(\frac{kT}{mc^2}\right)^2 + \dots . \tag{43}$$

Combining Eqs. (37), (42), and (43), we finally obtain

$$\mu_t(\nu) = \frac{1}{u\ell_t(\nu)} N\sigma_0 \frac{kT}{mc^2} \sum_n n a_n u^n [u - (n + 3) - \delta(u,T) + \dots] , \tag{44}$$

$$\delta(u,T) = \frac{kT}{10\ mc^2}\Big[(7n + 35)u^2 - (14n^2 + 84n + 137)u$$
$$+ (7n^3 + 42n + 60n - 9)\Big] . \tag{45}$$

We assume that ionization is complete and that it is permissible to neglect all absorptive processes other than free-free transitions. Then if we neglect electron degeneracy and insert numerical values, the usual expression for the free-free absorption coefficient in units of cm^{-1} becomes

$$\mu'_a(\nu) \simeq \mu'_{ff}(\nu) = \left[2.782 \frac{cm^{-5}(kev)^{-7/2}}{moles^2}\right] \frac{\rho^2(kT)^{-7/2}}{u^3} [1 - \exp(-u)]$$
$$\cdot \frac{\left(\sum_i \alpha_i Z_i\right)}{\left(\sum_i \alpha_i M_i\right)^2} \sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT, u), \tag{46}$$

where kT is in kilovolts and ρ is in g/cm^3. Here Z_i, M_i, $g_{ff}(Z_i^2/kT,u)$, and α_i are the atomic number, the atomic weight, the Gaunt factor for free-free transitions, and the mole fraction for element i, respectively. Using these same units in Eq. (44), substituting Eqs. (44) and (46) into the second of Eqs. (8), multiplying through by $\ell_t(\nu)$, and using Eq. (36), we get

$$\sum a_n X_n = 1; \tag{47}$$

$$X_n = A\left\{Bu^{(n-4)}[1 - \exp(-u)]\sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT,u) + nu^{(n-1)}[u - (n + 3) - \delta(u,T)]\right\}, \tag{48}$$

where

$$A = \left(7.843 \times 10^{-4} \frac{\text{cm}^2}{\text{mole-kev}}\right)\rho kT \frac{\sum_i \alpha_i Z_i}{\sum_i \alpha_i M_i};$$

$$B = \left[3.547 \times 10^3 \frac{(\text{kev})^{9/2}\text{cm}^3}{\text{moles}}\right]\frac{\rho}{(kT)^{9/2}\sum_i \alpha_i M_i}. \tag{49}$$

These equations can then be solved numerically to obtain the a_n. In doing this, we took a finite number of terms (usually 7 or 8 with values of n between -0.5 and 1.25) on the right-hand side of Eq. (36), evaluated the corresponding X_n of Eq. (48) for specific values of u between 0.1 and 30, and used Eq. (47) to make numerical fits for the a_n and hence $\ell_t(\nu)$ on an IBM 704 computer.

If we neglect the second order scattering term, $\delta(u,T)$, and combine Eqs. (8), (36), (44), and (46), we can also obtain the second order differential equation

$$\frac{d^2y}{du^2} = (1 - \frac{4}{u}) \frac{dy}{du} + B \frac{[(1 - \exp(-u)]}{u^5} \sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT,u)\ y - \frac{1}{Au}, \qquad (50)$$

$$y = u\ell_t(\nu).$$

This equation can then be solved by numerical integration. We decomposed it into two simultaneous first order differential equations as described by Margenau and Murphy (1943, p. 473) and used the Runge-Kutta method of numerical integration. Before discussing the results obtained by these methods, we derive some transformation equations which considerably decrease the quantity of results needed.

V. TRANSFORMATION EQUATIONS

When $\delta(u,T)$ in Eq. (44) is neglected, results $\rho_1^2\left[\ell_t(\nu_1)\right]_1$ for one temperature, T_1, and density, ρ_1, can be transformed to results $\rho_2^2\left[\ell_t(\nu_2)\right]_2$ corresponding to a new temperature, T_2, and density, ρ_2. We consider T_1 and T_2 fixed and transform between "corresponding densities," ρ_1 and ρ_2. By "corresponding densities," we mean that ρ_2 is chosen such that the ratio of absorptive to scattering contributions to $\ell_t(\nu)^{-1}$ for a given value of u are the same at ρ_2 and T_2 as they were at ρ_1 and T_1, then the

$\ell_t(\nu)$ versus u curves for the two sets of densities and temperatures have similar shapes. From Eqs. (36), (47), (48), and (49), we see that

$$\rho_2(\nu_2) = \rho_1 \left(\frac{T_2}{T_1}\right)^{9/2} \frac{\sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT_1, u)}{\sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT_2, u)} ;$$

$$\left[\ell_t(\nu_2)\right]_2 = \frac{\rho_1 T_1}{\rho_2(\nu) T_2} \left[\ell_t(\nu_1)\right]_1 ; \tag{51}$$

$$\nu_2 = \frac{T_2 \nu_1}{T_1} .$$

Combining these two equations, we get the more convenient relation for our purposes

$$\rho_2^2(\nu_2)\left[\ell_t(\nu_2)\right]_2 = \left(\frac{T_2}{T_1}\right)^{7/2} \frac{\sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT_1, u)}{\sum_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT_2, u)} \rho_1^2\left[\ell_t(\nu_1)\right]_1 ;$$

$$\nu_2 = \frac{T_2 \nu_1}{T_1} . \tag{52}$$

Similarly, one can calculate results for one mixture (subscripts 2 and j) from those for another mixture (subscripts 1 and i). In this case, it is best to choose $T_2 = T_1$ so the relative contribution of the second order scattering term, $\delta(u,T)$, is the same for both mixtures. We get

$$\rho_2^2(\nu)\Big[\ell_t(\nu)\Big]_2 = \left[\frac{\sum\limits_j \alpha_j M_j}{\sum\limits_i \alpha_i M_i}\right]^2 \left[\frac{\sum\limits_i \alpha_i Z_i}{\sum\limits_j \alpha_j Z_j}\right] \left[\frac{\sum\limits_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT,u)}{\sum\limits_j \alpha_j Z_j^2 g_{ff}(Z_j^2/kT,u)}\right] \rho_1^2\Big[\ell_t(\nu)\Big]_1, \quad (53)$$

$$\rho_2(\nu) = \rho_1 \left[\frac{\sum\limits_j \alpha_j M_j}{\sum\limits_i \alpha_i M_i}\right] \left[\frac{\sum\limits_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT,u)}{\sum\limits_j \alpha_j Z_j^2 g_{ff}(Z_j^2/kT,u)}\right],$$

$$T_1 = T_2 = T. \quad (54)$$

The frequency dependence of $\rho_2(\nu_2)$ and $\rho_2(\nu)$ is very small and can be eliminated by a slight adjustment of the results for $\rho_2^2(\nu)\Big[\ell_t(\nu_2)\Big]_2$ and $\rho_2^2(\nu)\Big[\ell_t(\nu)\Big]_2$, respectively, without introducing much error. If the appropriate average values are used for the Gaunt factors, Eqs. (51) to (54) also apply with the frequency dependence of ρ_2 removed and $\ell_t(\nu_2)$, $\ell_t(\nu_1)$, and $\ell_t(\nu)$ everywhere replaced with Λ_t. We find empirically from the results in the next section (in Table 1) that the value of u for which the Gaunt factors equal their appropriate average values increases from slightly less than 0.5 in the low density limit to about 7.5 in the high density limit. The exact value used isn't very critical, because the ratio of free-free Gaunt factors varies so slowly with frequency.

For frequencies above all absorption edges, the contribution to the absorption coefficient due to bound-free transitions, $\mu_{bf}'(\nu)$, has almost the same frequency dependence as the free-free contribution, $\mu_{ff}'(\nu)$.

Thus at low temperatures, one can include the bound-free contribution by making another transformation similar to those described above. In this case, we get

$$\rho_2(\nu) = \rho_1\left[1 + \frac{\mu'_{bf}(\nu)}{\mu'_{ff}(\nu)}\right]^{-1} ; \tag{55}$$

and

$$\begin{aligned} \rho_2^2(\nu)\left[\ell_t(\nu)\right]_2 &= \rho_1^2\left[\ell_t(\nu)\right]\left[1 + \frac{\mu'_{bf}(\nu)}{\mu'_{ff}(\nu)}\right]^{-1} , \\ T_1 = T_2 &= T, \end{aligned} \tag{56}$$

where for $\mu'_{bf}(\nu)/\mu'_{ff}(\nu)$ we use the approximate equation

$$\frac{\mu'_{bf}(\nu)}{\mu'_{ff}(\nu)} = \frac{2\ \mathrm{Ryd}}{kT} \frac{\sum\limits_{i,n} n^{-3}\alpha_i Z_i^4 g_{bfn}(Z_i^2/kT,u)\exp(Z_i^2\ \mathrm{Ryd}/n^2kT)}{\sum\limits_i \alpha_i Z_i^2 g_{ff}(Z_i^2/kT,u)} . \tag{57}$$

Here quantities with subscripts 2 and 1 refer to results with and without $\mu'_{bf}(\nu)$ included, respectively; $g_{bfn}(Z_i^2/kT,u)$ is the bound-free Gaunt factor for element i in initial bound state with principal quantum number n; and Ryd designates 1 Rydberg. These transformation equations also apply with Λ_t replacing $\ell_t(\nu)$ and ρ_2 constant when an average value is used for the slowly varying function $\mu'_{bf}(\nu)/\mu'_{ff}(\nu)$. Equations (55) to (57) are valid only for frequencies with energies greater than the ionization energies of all levels. Thus the integrated form of these equations applies only when this includes all frequencies important in determining Λ_t. The range of conditions is small for which this is true and also for which $\mu'_{bf}(\nu)$ is significantly large; however, this transformation is still useful for low temperatures, because the correct results for $\rho^2\Lambda_t$ and ρ always lie between those obtained with and without its usage.

Of course, for most of the relatively low temperature region for which bound-free and bound-bound absorption is important, scattering is not important. Then $\Lambda_t \simeq \Lambda = 1/\rho K$, where K is the Rosseland mean opacity. Even when scattering makes a significant contribution to K, one can usually make the approximation $\Lambda_t \simeq 1/\rho K'$, where K' is the opacity with the scattering contribution omitted, because $\mu_t(\nu)$ is of the order $\mu_s(\nu)kT/mc^2$.

VI. RESULTS

In view of the existence of these transform equations, there is no need to compute results for more than one element or for temperatures below that for which $\delta(u,T)$ in Eq. (44) becomes insignificant. This occurs at $\simeq$ 1 kev. Using the first method described in the last section, results were calculated for hydrogen at various densities in the temperature range 1 kev $\leq$ kT $\leq$ 16 kev.

As a check, some of the results obtained this way were used as initial starting values for $y = u\ell_t(\nu)$ in integrating inward on an IBM 704 to solve Eq. (50) by the Runge-Kutta method. The latter method gives more accurate results when applicable but consumes more computing machine time, even though a double precision code was used in obtaining results by the first method. When u is large, $y \simeq$ constant. Thus dy/du is very small, and its starting values are not needed with great accuracy. We used the equation

$$\left.\frac{dy}{du}\right|_{\text{large } u} \simeq \frac{1}{A}\left[\frac{1}{u - 4} - \frac{u}{(u - 4)^3}\right] . \qquad (58)$$

If the initial values of y were chosen slightly too large (or small), y became large in magnitude and positive (or negative) for small u. By this method, one could obtain accurate results only down to the values of u for which $\ell_t(\nu)$ began to decrease rapidly as u was decreased and approached the value $\left[\mu_a'(\nu)\right]^{-1}$. At kT = 1 kev, it was found by this differential equation method that, with the exception of very low frequencies, $\rho^2\ell_t(\nu)$ was equal to 2.522/u, 2.351/u, 2.324/u, and 2.321/u for the densities 10^{-7}, 10^{-9}, 10^{-11}, and 10^{-13} g/cm^3, respectively. Thus we conclude that

$$\lim_{\rho\to 0} \rho^2\ell_t(\nu) = \frac{b(kT)}{u} ; \qquad u \neq 0, \qquad (59)$$

where the constant b(kT) has the value 2.32 when kT = 1 kev. A comparison of results for different temperatures indicated that b(kT) has essentially the same temperature dependence as $\rho^2\Lambda_a$, the Rosseland mean of $\rho^2\left[\mu_a'(\nu)\right]^{-1}$.

The final results for $\rho^2(kT)^{-7/2}\Lambda_t$ in units of g^2 $\text{kev}^{-7/2}/\text{cm}^5$ are given in Table 1. It is expected that the numerical errors in these results are within 2, 3, 4, 6, and 8% for kT values of 1, 2, 4, 8, and 16 kev, respectively. For low densities, the error is probably less than this. The numbers in parentheses were obtained with the second order scattering term $\delta(u,T)$ of Eqs. (44) and (48) neglected.

The results obtained for $\rho^2\ell_t(\nu)$ in units of g^2/cm^5 are presented in Table 2. The error arising from the numerical methods which were used is less than 5% (usually much less) with the exception of the points $u = 0.1$, $\rho = 10^{-4}$ g/cm^3; $u = 0.2$, $\rho = 10^{-3}$ g/cm^3; $u = 0.4$, $\rho = 10^{-2}$ g/cm^3; and $u = 0.6$, $\rho = 10^{-2}$ g/cm^3 which have uncertainties of 40, 60, 20, and 10%, respectively.

The results given in both tables for $\rho \rightarrow \infty$ were simply obtained by setting $\ell_t(\nu) = \left[\mu_a'(\nu)\right]^{-1}$. Since Eq. (46) was used for $\mu_a'(\nu)$, degeneracy was ignored. Table 1 is not complete on the high density side for $kT \geq 4$ kev, because in this region the electrons are partially degenerate.

In obtaining the results given in Table 2, the second order term, $\delta(u,T)$, was neglected. Its maximum effect is to increase $\ell_t(\nu)$ by about 10 or 12% at the points $\rho = 1$ g/cm^3, $u = 30$ and $\rho = 10$ g/cm^3, $u = 30$; however its effect is usually much less than this. The approximate magnitude of the effect of $\delta(u,T)$ on $\rho^2\ell_t(\nu)$ is indicated in Figure 1 where

Table 1

Values for $\rho^2(kT)^{-7/2}\Lambda_t$ for Various Temperatures and Densities of Hydrogen*

ρ (g/cm^3)	kT (kev) 1	 2	 4	 8	 16
∞	158.	171.	181.	189.	195.
10^3			91.1 (85.4)	31.0 (26.9)	7.75 (6.34)
10^2		104.2 (101.6)	39.4 (37.0)	10.09 (9.09)	2.86 (2.53)
10^1	114.	51.2 (49.8)	13.73 (13.10)	3.61 (3.40)	1.422 (1.347)
10^0	63.6	19.1	4.74	1.68	0.982 (0.957)
10^{-1}	26.3	6.56	2.04	1.05	0.813
10^{-2}	9.11	2.58	1.156 (1.152)	0.834	
10^{-3}	3.35	1.34	0.867		
10^{-4}	1.57	0.910	0.755		
10^{-5}	0.989	0.764			
10^{-6}	0.782				
10^{-7}	0.700				
0	0.644	0.664	0.679	0.691	0.703

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*Entries in parentheses have been computed with the effect of the second order term $\delta(u,T)$ omitted. Λ_t is in units of cm.

Table 2

Values for $\rho^2 \ell_t(\nu)$ for Hydrogen at a Temperature of 1 kev

$u = \frac{h\nu}{kT}$	ρ (g/cm^3)								
	∞	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
0.1	2.02×10^{-3}	2.02×10^{-3}	2.02×10^{-3}	2.02×10^{-3}	2.02×10^{-3}	2.02×10^{-3}	4.00×10^{-1}	1.05×10^{1}	2.25×10^{1}
0.2	1.00×10^{-2}	1.00×10^{-2}	1.00×10^{-2}	1.00×10^{-2}	1.00×10^{-2}	4.00×10^{-2}	4.01×10^{0}	1.20	1.33
0.4	5.38	5.38	5.38	5.38	6.30	1.81×10^{0}	7.30	7.92×10^{0}	6.93×10^{0}
0.6	1.50×10^{-1}	1.50×10^{-1}	1.50×10^{-1}	1.50×10^{-1}	4.00×10^{-1}	4.55	6.76	5.62	4.67
0.8	3.19	3.19	3.19	3.19	1.62×10^{0}	6.05	5.81	4.31	3.50
1.0	5.84	5.84	5.84	6.40	3.53	6.58	4.98	3.49	2.81
1.2	9.71	9.71	9.71	1.34×10^{0}	5.55	6.61	4.33	2.93	2.34
1.4	1.51×10^{0}	1.51×10^{0}	1.51×10^{0}	2.54	7.31	6.41	3.82	2.53	2.01
1.6	2.23	2.23	2.25	4.35	8.67	6.13	3.41	2.22	1.76
1.8	3.16	3.16	3.36	6.62	9.67	5.82	3.07	1.98	1.56
2.0	4.35	4.35	4.86	9.25	1.03×10^{1}	5.50	2.80	1.78	1.41
2.2	5.79	5.79	7.00	1.21×10^{1}	1.07	5.20	2.57	1.62	1.28
2.4	7.63	7.63	9.50	1.48	1.10	4.93	2.37	1.49	1.17
2.6	9.82	9.82	1.25×10^{1}	1.76	1.11	4.67	2.20	1.38	1.08
2.8	1.24×10^{1}	1.28×10^{1}	1.59	2.00	1.11	4.44	2.05	1.28	1.01
3.0	1.55	1.61	2.00	2.22	1.10	4.22	1.92	1.19	9.40×10^{-1}
3.2	1.91	2.03	2.49	2.41	1.09	4.03	1.81	1.12	8.81
3.4	2.33	2.47	3.02	2.58	1.08	3.85	1.71	1.06	8.29
3.6	2.82	3.00	3.59	2.73	1.06	3.68	1.62	9.97×10^{-1}	7.83
3.8	3.37	3.60	4.19	2.84	1.04	3.53	1.54	9.45	7.42

Table 2, Continued

$u = \frac{h\nu}{kT}$	ρ (g/cm^3)								
	∞	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
4.0	4.00×10^1	4.30×10^1	4.81×10^1	2.95×10^1	1.03×10^1	3.39×10^0	1.47×10^0	8.98×10^{-1}	7.05×10^{-1}
4.5	5.95	6.35	6.32	3.13	9.77×10^0	3.08	1.31	7.99	6.27
5.0	8.48	8.90	7.73	3.23	9.30	2.82	1.18	7.19	5.64
5.5	1.17×10^2	1.21×10^2	8.98	3.27	8.84	2.61	1.08	6.54	5.13
6.0	1.57	1.58	1.00×10^2	3.28	8.42	2.42	9.94×10^{-1}	6.00	4.70
6.5	2.06	1.98	1.09	3.27	8.03	2.26	9.20	5.54	4.34
7.0	2.65	2.39	1.16	3.23	7.67	2.12	8.56	5.15	4.03
7.5	3.36	2.81	1.22	3.19	7.34	2.00	8.01	4.81	3.76
8.0	4.19	3.21	1.27	3.14	7.03	1.89	7.53	4.51	3.53
8.5	5.16	3.60	1.30	3.08	6.75	1.79	7.10	4.24	3.32
9.0	6.29	3.97	1.33	3.02	6.49	1.70	6.71	4.01	3.14
10.0	9.01	4.62	1.36	2.90	6.03	1.55	6.06	3.61	2.82
11.0	1.25×10^3	5.16	1.37	2.79	5.63	1.43	5.52	3.28	2.57
12.0	1.69	5.59	1.37	2.67	5.28	1.32	5.08	3.01	2.35
13.0	2.22	5.94	1.36	2.57	4.98	1.23	4.70	2.78	2.17
14.0	2.86	6.20	1.35	2.47	4.71	1.15	4.37	2.58	2.02
15.0	3.64	6.41	1.33	2.38	4.47	1.08	4.09	2.41	1.88
20.0	9.80	6.80	1.21	2.01	3.57	8.32×10^{-1}	3.09	1.81	1.41
25.0	2.15×10^4	6.70	1.10	1.74	2.99	6.79	2.48	1.45	1.13
30.0	3.98	6.25	1.00	1.53	2.58	5.74	2.07	1.21	9.42×10^{-2}

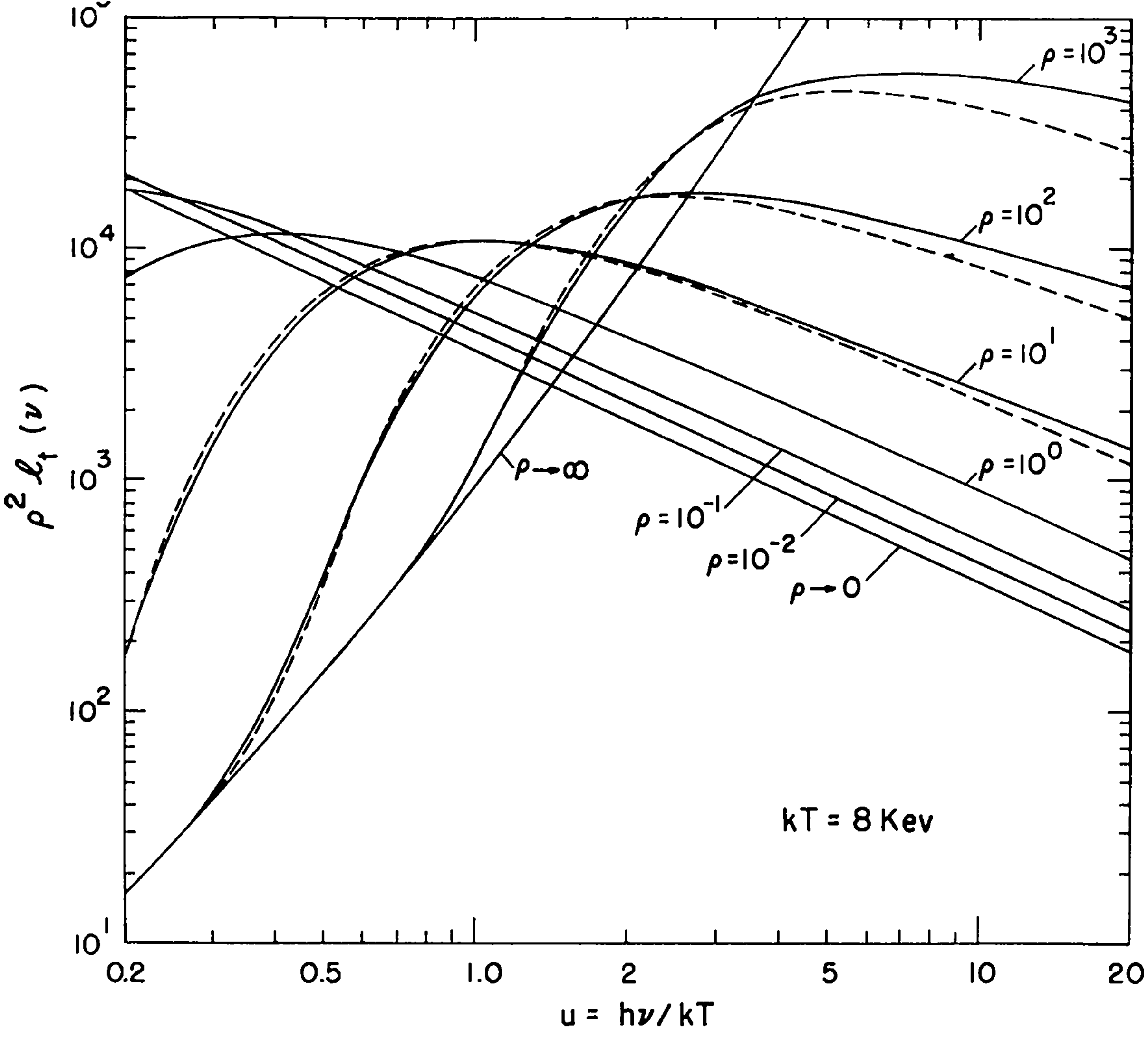

Figure 1 A plot of $c\rho^2$ times the photon relaxation time, $c^{-1}\ell_t(\nu)$, versus u for various densities and a temperature of 8 kev. Here $\ell_t(\nu)$ is in units of cm and ρ is in g/cm^3. Results are for hydrogen.

$\rho^2 \ell_t(\nu)$ versus u is plotted for various densities at a temperature of 8 kev. The results with and without $\delta(u,T)$ included are represented by solid and dashed curves, respectively. As one would expect from Eqs. (44), (45), and (48), it was found that for "corresponding densities" defined above Eq. (51), the magnitude of the contribution arising from $\delta(u,T)$ to either $\rho^2 \ell_t(\nu)$ or $\rho^2 \Lambda_t$ increases approximately linearly with temperature from zero at $kT = 0$. Thus in obtaining results for $\rho^2 \ell_t(\nu)$ or $\rho^2 \Lambda_t$ for new temperatures, one should apply Eqs. (51) and (52) to the results in Table 2 or Table 1 with $\delta(u,T)$ omitted and then add the appropriate correction, if significant, as determined from inspection of Figure 1 or Table 1.

In obtaining all results for $\ell_t(\nu)$ and Λ_t, the numerical values of Karzas and Latter (1958) for the free-free Gaunt factors were used. Their results are in good agreement with those of Berger (1956) where they overlap.

VII. DISCUSSION

Noting that $\mu_a'(\nu)$ given by Eq. (46) is proportional to ρ^2, while $\mu_t(\nu)$ is proportional to ρ (if the density dependence of $\ell_t(\nu_2)/\ell_t(\nu)$ is neglected), and that $\mu_a'(\nu)$ goes as about ν^{-3}, while $\mu_t(\nu)$ has a much weaker frequency dependence, we interpret our results as follows: At very high densities, $\mu_t(\nu)$ is negligible, and we see from the second of Eqs. (8) that $\ell_t(\nu) = \left[\mu_a'(\nu)\right]^{-1}$. Then $\rho^2 \Lambda_t = \rho^2 \Lambda_a$, and, insofar as electron degeneracy can be neglected, $\rho^2\Lambda_t$ and $\rho^2 l_t(\nu)$ are independent of density. As density is lowered, $\mu_t(\nu)$ makes a signficant contribution first at high frequencies and then at lower and lower frequencies. Since the correct physical solution has positive a_n and n for the major terms in Eq. (36), we see from the dominant part of Eq. (44) that $\mu_t(\nu)$ is negative for small u. When the density is low, $\mu_a'(\nu)$ is small except for very small frequencies. Thus for low densities, if $l_t(\nu)$ is to be positive when $\mu_t(\nu)$ is negative, we see from Eqs. (8) that $\mu_t(\nu)$ must be very small in magnitude. As seen from Eq. (9), this occurs when $ul_t(\nu)$ is very nearly independent of frequency. It must also be very large and proportional to ρ^{-2} because $\mu_a'(\nu)$ and $\mu_t'(\nu)$ are nearly cancelling and $\mu_a'(\nu)$ is proportional to ρ^2. Thus we obtain the result expressed by Eq. (59). Since this equation applies for all but negligibly small frequencies, we can integrate Eq. (24) and obtain

$$\lim_{\rho \to 0} \rho^2\Lambda_t = 0.2776b(kT). \qquad (60)$$

As seen from Eqs. (3), when the temperature is increasing (or decreasing), the number of photons of every frequency must increase (or decrease) if a distribution close to a Planck distribution is to be maintained. The total number of photons can be changed only by absorptive and emissive processes, not by scattering. Thus in the low density region for which Eqs. (59) and (60) apply, the controlling factor is the rate of production or destruction of photons, which is proportional to

$\mu_a'(\nu)$. This is the physical reason why $\rho^2 \ell_t(\nu)$ and $\rho^2 \Lambda_t$ again become independent of density at low densities. Compton scattering redistributes the photons. This causes the $\rho^2 \ell_t(\nu)$ versus u curve to be very different in shape from the $\rho^2 \left[\mu_a'(\nu)\right]^{-1}$ versus u curve, but it has a maximum effect on $\rho^2 \Lambda_t$ of only reducing it to about 1/250 times its high density value $\rho^2 \Lambda_a$.

This is completely contrary to the result one would obtain, for instance, if he assumed that the radiation had a Planck distribution at a temperature $T_r \neq T$. With this assumption, the scattering contribution on the right-hand side of Eq. (4) becomes independent of absorption. It then goes as ρ, while $\mu_a'(\nu)$ goes as ρ^2. Thus at very low densities, absorption can be neglected, and one gets a time of equipartition of energy between radiation and matter which is proportional to $1/\rho$ rather than $1/\rho^2$ as obtained by our method. However, as just pointed out below Eq. (60), it is inconsistent in a time-dependent problem to assume that a Planck distribution is maintained by scattering alone.

For high densities, this method leads one to obtain a time of equipartition of energy between radiation and matter equal to $\left[c\bar{\mu}_a'(\nu)\right]^{-1}$ rather than $c^{-1}\Lambda_a$ as obtained by our method, where $\bar{\mu}_a'(\nu)$ and Λ_a are the Rosseland means of $\mu_a'(\nu)$ and $\left[\mu_a'(\nu)\right]^{-1}$, respectively. $\left[c\bar{\mu}_a'(\nu)\right]^{-1}$ is always less than $c^{-1}\Lambda_a$ unless $\mu_a'(\nu) = \text{const}$. In the case of free-free absorption by hydrogen at 1 kev, $\left[c\bar{\mu}_a'(\nu)\right]^{-1} \simeq c^{-1}\Lambda_a/25$. For high densities, the error in assuming a Planck distribution at $T_r \neq T$ is seen by inspection of the absorptive term on the right-hand side of Eq. (4). We see that the time rate of change of the radiative intensity is proportional to $\mu_a'(\nu)[B(\nu,T) - I(\nu)]$. Thus when $I(\nu)$ and $B(\nu,T)$ are nearly equal, small deviations in $I(\nu)$ from a Planck distribution are very important. Now we expect $I(\nu)$ to be most near to $B(\nu,T)$ at the frequencies for which the interaction between radiation and matter is large, i.e., $\ell_t(\nu)$ small. This effect is taken into account in Eq. (19), whereas it is not by assuming $I(\nu)$ has a Planck distribution at T_r. The reason for pointing out the fallacy of this approach in dealing with an optically thick medium is that one might be tempted to use it, because it makes the energy exchange rate between radiation and matter easy to compute. Also one might expect it to be valid by analogy to the situation where ions and electrons are not in equilibrium. In the latter case, it is usually a good approximation to assume that ions and electrons each have a Maxwellian distribution, but at different temperatures, because these particles exchange energy most readily with other particles of the same kind. On the other hand, photons interact only with the matter, not with each other, and for this reason tend toward a Planck distribution only at the matter temperature (the electron temperature when ions and electrons are not in equilibrium).

It is interesting to note that the form given by Eq. (60) is perfectly general in the low density limit regardless of the particular form of the frequency dependence of $\mu_a'(\nu)$ or whether or not the expansions leading to Eq. (44) for $\mu_t(\nu)$ are valid. However, if the driving term $\mu_a'(\nu)$ went as a positive power of ν, Eq. (59) would hold for all but very large rather than small values of u. In this case, the n's of the dominant terms in Eq. (36) would be small in magnitude and negative, so $\mu_t(\nu)$ would be positive for small u and negative for large u, and $\ell_t(\nu)$ would have the limiting value $\left[\mu_a'(\nu)\right]^{-1}$ at large rather than small u.

We consider now the validity, for various conditions, of the approximations made in obtaining our solution to the radiative transport equation.

Our results for radiative intensity, energy density, and flux apply in the system stationary with respect to the material medium. In a time-dependent problem, we expect the local macroscopic velocity $\underset{\sim}{V}$ of the matter to vary with space and time. Thus, in contrast to the method outlined by Thomas (1930), the system in which our results apply is not in general an inertial system. This introduces an error in the higher than zeroth order terms in our expression (19) for the intensity. The condition for the validity in this respect of our approach is that the magnitude of the variation of $\underset{\sim}{V}/c$ must be small compared with unity over a space or time interval such that T and/or ρ change by a large fraction of themselves. Due to the large value of c, this condition is probably always satisfied.

Since it has been emphasized that the solution obtained here is valid only for an optically thick medium, we should specify more precisely what "optically thick" means. When absorption is dominant,

$\Lambda = \Lambda_t$, and it is clear that in order to be optically thick, a medium must not change appreciably in temperature, density, or composition within a distance of a few mean free paths in any direction. The meaning is not so clear when scattering is significant. We consider the extreme case where scattering is very important. Since then $\Lambda_t \gg \Lambda = \Lambda_s$, we investigate the time-dependent part of the solution to the radiative transport equation. This indicates that the radiation spectrum at position P at time t_o is determined in the immediately prior time interval equal to approximately $3c^{-1}\Lambda_t$, where the choice of the exact factor 3 is somewhat arbitrary. If it assumed that the boundary of a region with appreciably different characteristics with regard to temperature, density, or composition is perpendicular to the x axis at a distance x_o from P, then in order that our solution be valid at P and t_o, it is necessary that the following condition be satisfied:

$$\left|\Delta'\mathcal{E}(P,t_o) - \Delta\mathcal{E}(P,t_o)\right| \ll \mathcal{E}(P,t_o), \tag{61}$$

where $\Delta'\mathcal{E}(P,t_o)$ and $\Delta\mathcal{E}(P,t_o)$ are the portions of the total radiation energy density $\mathcal{E}(P,t_o)$ calculated to originate from $x \geq x_o$ when the presence of the boundary at x_o is taken into account and when it is ignored, respectively. We assume that condition (61) is satisfied for values of x_o such that $\Delta\mathcal{E}(P,t_o)/\mathcal{E}(P,t_o)$ is small. In order to obtain an approximate value for $\Delta\mathcal{E}(P,t_o)/\mathcal{E}(P,t_o)$, we solve the inverse problem of obtaining the probability that photons originally at P at time t_o will be at $x > x_o$ at time $t_o + 3c^{-1}\Lambda_t$. An approximate value for this, in turn, can be obtained from the well-known solution to random flight problem given, for instance, by Eq. (87), page 14, of Chandrasekhar (1943). Since photons travel with velocity c and have flight lengths $\Lambda = \Lambda_s$, the number of random flights taken by a photon in time interval $3c^{-1}\Lambda_t$ is $3\Lambda_t/\Lambda_s$. If this number is large (it always is when $\Lambda \simeq \Lambda_s$), the probability that photons at P at time t_o will be a distance $\underset{\sim}{r}$ from P at time $t_o + 3c^{-1}\Lambda_t$ is

$$W(\underset{\sim}{r}, t_o + 3c^{-1}\Lambda_t) = (2\pi\Lambda_t\Lambda_s)^{-3/2} \exp\left(- \frac{r^2}{2\Lambda_t\Lambda_s}\right). \tag{62}$$

Thus

$$\frac{\Delta\mathcal{E}(P,t_o)}{\mathcal{E}(P,t_o)} \simeq 1 - \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{x_o} dx W(\underset{\sim}{r}, t_o + 3c^{-1}\Lambda_t)$$

$$= 1 - \int_{-\infty}^{x_o} dx (2\pi\Lambda_t\Lambda_s)^{-1/2} \exp\left(- \frac{x^2}{2\Lambda_t\Lambda_s}\right). \tag{63}$$

This gives a value 0.05 for $\Delta\mathcal{E}(P,t_o)/\mathcal{E}(P,t_o)$ when $x_o = 1.65(\Lambda_t\Lambda_s)^{1/2}$. If Λ_s is replaced with Λ, this goes smoothly into the approximate result expected when $\Lambda = \Lambda_t$. Thus it is probably quite accurate to say that the solution to the radiative transport equation obtained in this report is valid a distance one or two times $(\Lambda\Lambda_t)^{1/2}$ from the boundary of a region of appreciably different characteristics, and an optically thick medium is one which is a few multiples of $(\Lambda\Lambda_t)^{1/2}$ or greater in thickness. We note that in reaching this conclusion, the value of the rate $\partial\mathcal{E}/\partial t$ did not enter. Thus it also applies to the time-independent situation treated, for instance, in S.

Since the interaction of the radiation with the matter is only through Compton scattering and free-free transitions (in treating the latter, the nuclei are assumed stationary), our basic assumption that the matter has distribution functions characteristic of thermal equilibrium really only applies to the electrons. For the total radiation, the criterion for the validity of this assumption is that $t_c \ll c^{-1}\Lambda_t$, where t_c is the relaxation time for electrons given by Eq. (5-26), page 78 of Spitzer (1956). This criterion is met. For, although the curves in Figure 2 are only approximate in the region of electron degeneracy as indicated by dashed lines, we see that the curve $c^{-1}\Lambda_t/t_c = 10$ lies somewhat above the curve $\Lambda_c/\Lambda = 10$, and much above the curve $E_m/\mathcal{E} = 10$, which indicate the approximate upper boundaries of the regions for which radiation flux and radiation energy density are significant. Here E_m is the kinetic energy density of the matter and Λ_c is the equivalent mean free path arising from electronic heat conduction, i.e., $\Lambda_c = (\rho K_c)^{-1}$, where K_c is the conductive opacity treated by Marshak (1940), Lee (1950), and Mestel (1950).[3]

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[3] Unfortunately, we used K_c in S to designate the opacity due to Compton scattering.

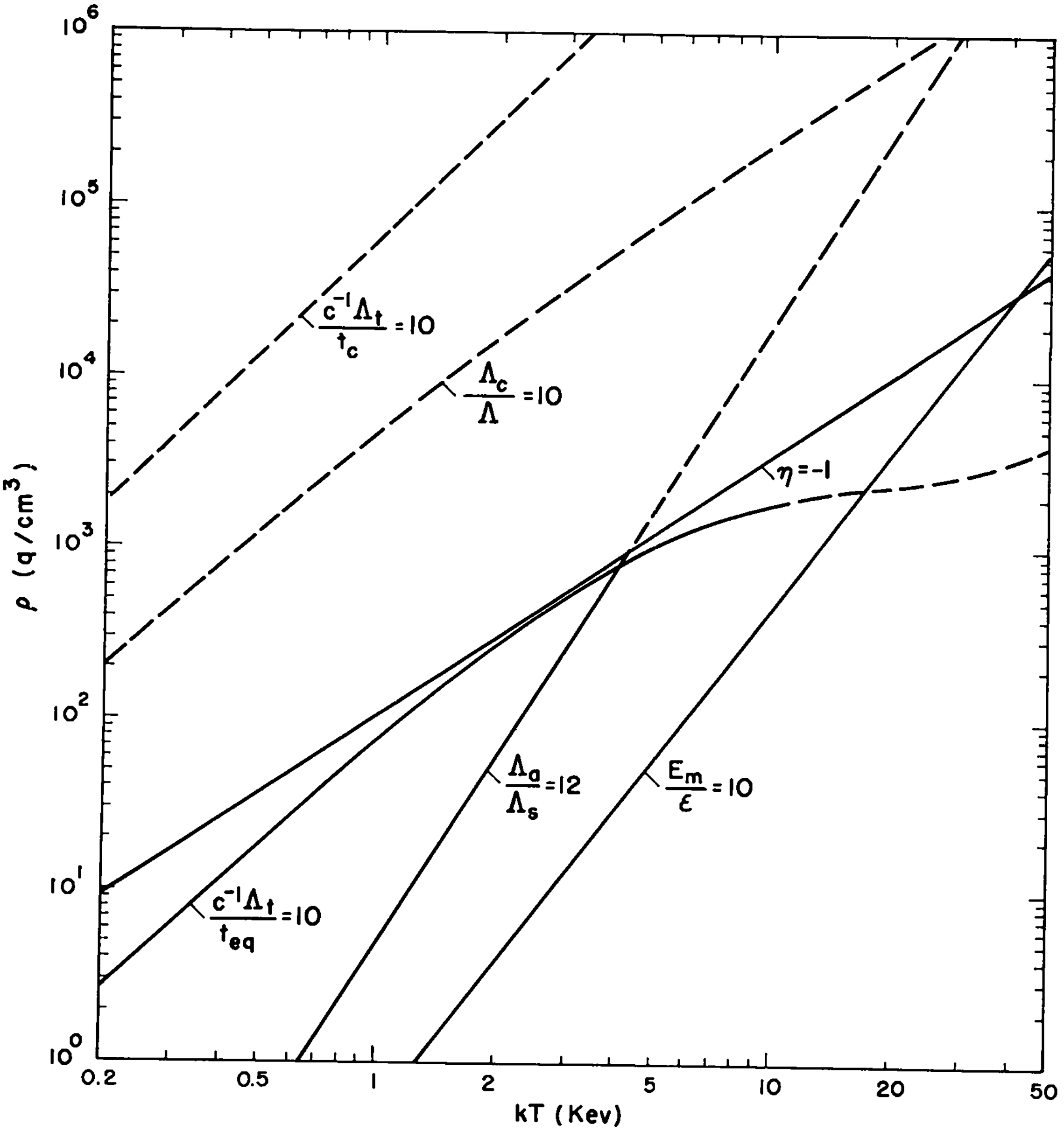

Figure 2 Regions of density and temperature for which various approximations apply. In treating the radiation, it can be assumed that the electrons are in thermal equilibrium with themselves below the curve $c^{-1}\Lambda_t/t_c = 10$ and are also in equilibrium with the ions below the curve $c^{-1}\Lambda_t/t_{eq} = 10$. Below the curve $\Lambda_a/\Lambda_s = 12$, a simplification in the time-dependent expression for the flux is valid. Electron degeneracy can be neglected below the curve $\eta = -1$. The radiative energy density and flux become relatively unimportant above the curves $E_m/\mathcal{E} = 10$ and $\Lambda_c/\Lambda = 10$, respectively.

Below the curve $c^{-1}\Lambda_t/t_{eq} = 10$, where t_{eq} is the electron-ion time of equipartition of energy given by Eq. (5-31), page 80 of Spitzer (1956), we can assume electrons and ions to be in local thermodynamic equilibrium with each other. Above this curve, the T in our radiation equations refers to the electron temperature.

Below the curve $\Lambda_a/\Lambda_s = 12$ in Figure 2, the approximation to the flux expressed by Eq. (29) is applicable. Usually the term proportional to Λ^2 can be omitted and Λ replaced with Λ_s. We designate as F' the first order time-dependent correction to F, i.e., the part of the flux proportional to $\partial T/\partial t$ and $\partial \rho/\partial t$; then along the curve $\Lambda_a/\Lambda_s = 12$, use of Eq. (29) results in values for F' which are too small in magnitude by about 25%. Along a line through the points $\rho = 0.9$ g/cm^3, kT = 0.5 kev; and $\rho = 25$ g/cm^3, kT = 1 kev, corresponding to $\Lambda_t/\Lambda_a \simeq 0.83$, use of Eq. (35) causes one to overestimate the magnitude of F' by about the same amount. In the region for which Eq. (29) applies, $F'/(F - F')$ and $\mathcal{E}'/(\mathcal{E} - \mathcal{E}')$, where $\mathcal{E}'$ is the time-dependent correction to the energy density given by the second term in Eq. (22), usually have about equal magnitudes, whereas in most of the region covered by Eq. (35), $F'/(F - F')$ is significantly larger than $\mathcal{E}'/(\mathcal{E} - \mathcal{E}')$ in magnitude. The reason for this is as follows: The Rosseland mean of a quantity which goes as u^n is roughly equal to that quantity evaluated at $u = n + 4$. The products of ℓ's occurring in the large parentheses of Eq. (35) go as considerably higher powers of u than $\ell(\nu)$ when absorption is dominant. Thus they are large relative to Λ^2, the rough approximate measure of the comparative importance of $\mathcal{E}'$.

Above the curve $\eta = -1$, electron degeneracy becomes important. Neglecting degeneracy along this curve causes one to underestimate the magnitude of F' by about 20 or 25% and Λ or Λ_t by about half that amount. Here η is the usual chemical potential multiplied by $1/kT$. It is equal to the α in S minus mc^2/kT.

The curves in Figure 2 all pertain to hydrogen. From a study of the expressions for the various quantities involved, we conclude that for an element with atomic number Z and atomic weight M, these curves should be shifted approximately as follows: Considering points at the same temperature for both elements, we shift down the curves $c^{-1}\Lambda_a/t_c \simeq c^{-1}\Lambda_t/t_c = 10$ and $\Lambda_a/\Lambda_s = 12$, so they pass through new densities, ρ_2, related to the old densities, ρ_1, by the equation $\rho_2 = M\rho_1/Z^2$. For the curves $\Lambda_c/\Lambda = 10$ and $\eta = -1$, the new densities should be equal to $M\rho_1/Z$. In the case of the curve $E_m/\varepsilon = 10$, $\rho_2 = 2M\rho_1/(Z + 1)$. The curve $c^{-1}\Lambda_t/t_{eq} = 10$ should remain unchanged except that the bend resulting when Λ_t departs appreciably from Λ_a should occur at a higher temperature, $T_2 \simeq \left(Z^2/M\right)^{2/5} T_1$.

It is interesting to consider the conditions under which matter is most likely to run away from the radiation, i.e., T becomes much greater than the average radiation temperature T_r, defined by the second of Eqs. (26). At high densities, $c^{-1}\Lambda_t = c^{-1}\Lambda_a \simeq A\rho^{-2}T^{3.6}$, where A is a constant. At low densities, when Eqs. (59) and (60) apply, $\Lambda_t \simeq \Lambda_a/250$. Thus from Eq. (23) and the second of Eqs. (25) and (26), we get

$$\frac{\partial \mathcal{E}}{\partial t} \simeq 4A^{-1} a\rho^{2} T^{0.4} \frac{\Delta T}{T} ; \qquad \text{high densities,}$$

$$\frac{\partial \mathcal{E}}{\partial t} \simeq 10^{3} A^{-1} a\rho^{2} T^{0.4} \frac{\Delta T}{T} ; \qquad \text{low densities,} \tag{64}$$

in which $\Delta T = (T - T_r)$. Since the rate of energy production per unit volume by the matter generally goes as a high power of temperature (until some very large temperature is reached) and as ρ^2, it appears that runaway is most likely to occur at high temperatures and fairly high densities.[4] However, the effect of runaway is largest at low densities and high temperatures when most of the energy at equilibrium is in the

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[4] Of course, the total rate R of energy transfer per unit volume from matter to radiation equals $\partial\mathcal{E}/\partial t + \underset{\sim}{\nabla} \cdot \underset{\sim}{F}$. We have assumed that $\partial\mathcal{E}/\partial t \gg \underset{\sim}{\nabla} \cdot \underset{\sim}{F}$. In determining this rate when $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} \gtrsim \partial\mathcal{E}/\partial t$, one should include in Eq. (19) for $I(\nu,\underset{\sim}{s})$ the second order term $\ell_2(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla} \cdot \left[\ell(\nu)\underset{\sim}{s} \cdot \underset{\sim}{\nabla}B(\nu,T)\right] \equiv \epsilon(\nu)$ of Eq. (S,12). For $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} \gg \partial\mathcal{E}/\partial t$, this leads (if the slight angular dependence of $\ell_2(\nu)$ is neglected) to $R(\nu)$ and R equal to the right-hand sides of Eqs. (25) with $\ell_2(\nu)$ and Λ_2' replacing $\ell_t(\nu)$ and Λ_t, respectively, where $\Lambda_2' = \int_0^\infty \epsilon(\nu)d\nu \Big/ \int_0^\infty \left[\epsilon(\nu)/\ell_2(\nu)\right]d\nu$. At high densities, $\ell_2(\nu) = \ell_t(\nu) = \left[\mu_a'(\nu)\right]^{-1}$, while at low densities, $\ell_2(\nu) \sim \ell(\nu) \ll \ell_t(\nu)$. When $\ell_2(\nu)$ depends strongly on ν, Λ_2' is somewhat larger than the Rosseland mean of $\ell_2(\nu)$. Thus if $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} \gtrsim \partial\mathcal{E}/\partial t$, R is less for high densities and greater for low densities than it is when $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} \ll \partial\mathcal{E}/\partial t$ for the same ΔT.

radiation field. Furthermore, the densities for which the first of Eqs. (64) applies are unlikely for $kT \gtrsim 8$ kev.

Actually, we have been implicitly assuming that the nuclei and electrons are in equilibrium, i.e., that we are dealing with the region beneath the curve $c^{-1}\Lambda_t/t_{eq} = 10$ in Figure 2. Above that curve, as seen from Figure 2, $\mathcal{E}$ is usually relatively small. Hence $\mathcal{E}' = -4\pi c^{-2}\Lambda_t \partial B(T)/\partial t$ is usually not too important. However, radiative flux is usually an important means of energy transport in this region, so the time-dependent correction F' to the flux is important. Since F' is generally negative, the energy flow rate is less than one might expect. In this region above the curve $c^{-1}\Lambda_t/t_{eq} = 10$, we should distinguish between the case where the processes by which energy is being generated involve a significant amount of gamma emission and where they do not. In the latter case, the energy generated first goes into kinetic energy of the nuclei; then it is possible for the nuclei to run away from the electrons. On the other hand, in the case of gamma emission, the energy is quickly converted into electron kinetic energy, principally by Compton collisions. This tends to make the electron temperature lead the ion temperature. Thus when gamma emission is dominant, the nuclei cannot run away from the electrons. One might expect the radiation to lead the electrons in this case. At the high frequencies corresponding to the gamma radiation, it does, but the remainder of the radiation lags the electrons as indicated by Eq. (21).

Since it appears that it would be desirable to have results for higher temperatures than those covered in Table 1 and perhaps for higher densities as well, we look into the possibility of obtaining them. The condition for the validity of the expansions leading to Eq. (44) for $\mu_t(\nu)$ is

$$2\gamma'_{\text{max}} \equiv 2\left(\frac{1+\beta}{1-\beta}\right)^{1/2} \frac{h\nu}{mc^2} \equiv \left[E + 1 + \left(E^2 + 2E\right)^{1/2}\right] \frac{2kT}{mc^2}\, u < 1, \qquad (65)$$

where E is the electron kinetic energy in units of mc^2. Choosing $E = 2kT/mc^2$, a value greater than the kinetic energy of most electrons, we find that the values of u for which $2\gamma'_{\text{max}} = 1$ are 24.9, 11.2, 4.86, and 2 for kT = 8, 16, 32, and 64 kev, respectively. The values of u important in determining Λ_t vary with density. At high densities, the integrand in Eq. (24) has a maximum at $u \simeq 7$, whereas for very low densities where Eq. (59) applies, it has a maximum at $u \simeq 3$. However, in the low density limit, the size of the constant b(kT) is determined primarily at lower frequencies where $\mu_t(\nu)$ is negative and there is near cancellation between $\mu'_a(\nu)$ and $\mu_t(\nu)$. This is substantiated by the fact that the inclusion of $\delta(u,T)$ had essentially no effect on the results calculated for $\rho^2 \ell_t(\nu)$ and $\rho^2\Lambda_t$ at low densities when kT = 8 and 16 kev and also by the fact noted at the end of section V that at low densities the Gaunt factors equal their appropriate average values when evaluated at u slightly less than 0.5. Hence, the method used here to obtain results for Λ_t is probably valid until $kT \simeq 64$ kev as long as $\rho \leq 10^3$ g/cm^3. This is the upper density for which Eq. (60) approximately holds when

$kT = 64$ kev. Accurate Gaunt factors are not available for 16 kev $< kT < 64$ kev; however, no more than a 50% error is likely to arise if we obtain results for this region of temperature and density by applying Eqs. (51) and (52) to the results for $kT = 16$ kev and choose the ratio of Gaunt factors equal to unity. Very rough estimates of Λ_t can also be obtained for higher densities and temperatures by use of these transformation equations.

For temperatures below about 12 kev, the condition $2\gamma'_{max} < 1$ applies for all frequencies important in determining Λ_t for any density. In this temperature region, the only difficulty at high densities is in the proper handling of electron degeneracy and screening effects. Some work has been done on computing the free-free absorption coefficient at high and moderately high densities by Tsao (1954), Zirin (1954), and Green (1958, 1960). In the discussion section of S, we outlined the changes to be made in the calculation of $\mu_s(\nu)$ when electron degeneracy is taken into account. They are the same for $\mu_t(\nu)$. For conditions at which positron concentration is negligible, the only change is that we replace $N(P)P^2dP$ in the integral equation (S,25) for $\mu_s(\nu)$ and Eq. (37) for $\mu_t(\nu)$ with

$$q(E,\eta,T,u - u_2)dE$$

$$= \frac{(E + 1)(E^2 + 2E)^{1/2}dE}{4\pi^3\lambda_0^3\left[\exp(-\eta + E/T') + 1\right]\left[\exp(\eta - E/T' - u + u_2) + 1\right]}, \qquad (66)$$

where, as before, E and T' designate the electron kinetic energy and temperature in units of mc^2, and λ_o is the Compton wavelength given by Eq. (S,41). As pointed out in S, the integrations are then very difficult to perform as a result of the quantity $(u - u_2)$ occurring in q; however, the average contribution of this quantity is probably very small until very high temperatures are reached. If we set $(u - u_2) = 0$ in Eq. (66), there is essentially no increase in work in calculating results for $\mu_t(\nu)$ in the region of electron degeneracy, except that one must take into account the effect of degeneracy and screening on $\mu_a'(\nu)$ because $\mu_t(\nu)$ depends on $\mu_a'(\nu)$ through the factor $\left[1 - \nu_2 \ell_t(\nu_2)/\nu \ell_t(\nu)\right]$ in Eq. (9).

In the case of $\mu_s(\nu)$, there is no increase in difficulty in obtaining results when the electrons are degenerate if we set $(u - u_2) = 0$ in Eq. (66), with the exception that one must, of course, calculate results for each value of η. However, as pointed out in the discussion section of S, the effect on $\mu_s(\nu)$ due to electron motion is small except at very low frequencies. Thus when the electrons are degenerate, it should be a fairly good approximation in obtaining results for the opacity due to Compton scattering simply to multiply the results obtained in S with degeneracy neglected by the factor

$$Q(\eta,T) = \left\{\int_0^\infty \frac{(E+1)(E^2+2E)^{1/2}dE}{\left[\exp(-\eta + E/T') + 1\right]\left[\exp(\eta - E/T') + 1\right]}\right\} \cdot \left\{\int_0^\infty \frac{(E+1)(E^2+2E)^{1/2}dE}{\left[\exp(-\eta + E/T') + 1\right]}\right\}^{-1} . \qquad (67)$$

Results for $Q(\eta,T)$ corresponding to several values of η and kT are presented in Table 3.

Table 3

Values for $Q(\eta,T)$ for Various Temperatures and Degeneracy Parameters η

kT (kev)	η							
	-3	-1	0.4	1.6	2.8	4	6	8
0	0.983	0.897	0.737	0.569	0.432	0.336	0.238	0.183
8	0.983	0.898	0.741	0.575	0.440	0.345	0.248	0.193
32	0.984	0.903	0.751	0.592	0.460	0.368	0.273	0.217
125	0.985	0.915	0.781	0.637	0.514	0.425	0.327	0.265

The values of the relaxation time $c^{-1}\Lambda_t$ determined from Table 1 are so small that we conclude that it is fairly unlikely that in any astrophysical problems dealing with optically thick mediums, the radiation fails to remain in local thermodynamic equilibrium with the matter. This is certainly true if the density and temperature are nearly uniform over large portions of the star under consideration. However, during the rapid stages of stellar evolution, we expect fairly steep gradients of ρ, T, and the local macroscopic velocity $\underset{\sim}{V}$ to exist. As pointed out below Eqs. (64), the rate of energy generation by nuclear processes generally goes as a high power of T and as ρ^2. Thus in small regions in which ρ and T happen to be appreciably higher than in neighboring regions, the critical values of ρ and T for which some nuclear process proceeds very rapidly may be reached. Possibly in some instances, these small sections of the star will then attain very high matter temperatures before cooling by expansion and outflow of energy flux. The probability of this occurring is enhanced if the time rates of change are large enough that the radiation lags the matter appreciably. For then the portion of energy which goes into the matter increases, thus increasing the matter temperature and rate of energy production, while the outflow of radiation flux is less for the same space derivatives of ρ and T than it would be if radiation and matter were in equilibrium. Perhaps in some cases, this is the way in which the very high temperatures needed for synthesis of such elements as iron are reached, rather than by attaining a very high temperature over large sections such as the whole stellar interior.

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